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The field theoretic renormalization group (RG) and the operator-product expansion are applied to the model of a transverse (divergence-free) vector quantity, passively advected by the “synthetic” turbulent flow with a finite (and not small) correlation time. The vector field is described by the stochastic advection-diffusion equation with the most general form of the inertial nonlinearity; it contains as special cases the kinematic dynamo model, linearized Navier-Stokes (NS) equation, the special model without the stretching term that possesses additional symmetries and has a close formal resemblance with the stochastic NS equation. The statistics of the advecting velocity field is Gaussian, with the energy spectrum $E(k) \propto k^{1-\varepsilon}$ and the dispersion law $\omega \propto k^{-2+\eta}$, k being the momentum (wave number). The inertial-range behavior of the model is described by seven regimes (or universality classes) that correspond to nontrivial fixed points of the RG equations and exhibit anomalous scaling. The corresponding anomalous exponents are associated with the critical dimensions of tensor composite operators built solely of the passive vector field, which allows one to construct a regular perturbation expansion in ε and η ; the actual calculation is performed to the first order (one-loop approximation), including the anisotropic sectors. Universality of the exponents, their (in)dependence on the forcing, effects of the large-scale anisotropy, compressibility, and pressure are discussed. In particular, for all the scaling regimes the exponents obey a hierarchy related to the degree of anisotropy: the more anisotropic is the contribution of a composite operator to a correlation function, the faster it decays in the inertial range. The relevance of these results for the real developed turbulence described by the stochastic NS equation is discussed.

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I. INTRODUCTION

It has become a commonplace to complain that theoretical understanding of turbulence remains the last unsolved problem of classical physics. Of course, the concept of turbulence refers to a great deal of disparate physical situations (“almost as varied as in the realm of life,” Ref. [1], p. 1) and any exhaustive and ultimate “theory of turbulence,” of course, can hardly ever be established. There is, however, a classical “list” of phenomena (or, rather, classes of phenomena) that represent and illustrate the main features of turbulence: existence and stability of solutions of hydrodynamics equations, convective turbulence, (in)stability of laminar flows and origin of turbulence, and so on. Those topics, which are of great practical and conceptual importance, have always remained the focus of attention for theoreticians. One of them is the fully developed (homogeneous, isotropic, inertial-range) hydrodynamical turbulence. Detailed description of this concept and the bibliography of this old but still open subject can be found in the classical monographs [1–3].

Turbulent flows that occur in various liquids or gases at very high Reynolds numbers reveal a number of general aspects (cascades of energy or other conserved quantities, scaling behavior with apparently universal “anomalous exponents,” intermittency, statistical conservation laws, and so on), which support the hopes that those phenomena can be explained within a self-contained and internally consistent theory. Recent developments in this area are presented and summarized in Ref. [4].

The most remarkable features of developed turbulence are encoded in the single term of intermittency. This concept has no rigorous definition within the classical probabilistic theory; an excellent introduction can be found in Ref. [5] and Chap. 8 Ref. [1]. Roughly speaking, intermittency means that statistical properties (for example, correlation or structure functions of the turbulent velocity field) are dominated by rare spatiotemporal configurations, in which the regions with strong turbulent activity have exotic (fractal) geometry and are embedded into the vast regions with regular (laminar) flow.

In the turbulence, such a phenomenon is believed to be related to strong fluctuations of the energy flux. Therefore, it leads to deviations from the predictions of the celebrated Kolmogorov-Obukhov (KO) phenomenological theory [1–3]. Such deviations, referred to as “anomalous” or “non-dimensional” scaling, manifest themselves in singular (arguably power-like) dependence of correlation or structure functions on the distances and the integral (external) turbulence scale L . The corresponding exponents are certain nontrivial and nonlinear functions of the order of the correlation function, the phenomenon referred to as “multiscaling.”

Within the framework of numerous semiheuristic models the anomalous exponents are related to statistical properties of the local dissipation rate, the fractal (Hausdorff) dimension of structures formed by the small-scale turbulent eddies, the characteristics of nontrivial structures (vortex filaments), and so on; see Refs. [1–3] for a review and further refer-

ences. The common drawback of such models is that they are only loosely related to underlying hydrodynamical equations, involve arbitrary adjusting parameters and, therefore, cannot be considered to be the basis for construction of a systematic perturbation theory in certain small (at least formal) expansion parameter; see, e.g., the remark in Ref. [6]. Thus serious doubts remain about the universality of anomalous exponents and the very existence of deviations from the KO theory.

The term “anomalous scaling” reminds of the critical scaling in models of equilibrium phase transitions. In those, the field theoretic methods were successfully employed to establish the existence of self-similar (scaling) regimes and to construct regular perturbative calculational schemes (the famous ε expansion and its relatives) for the corresponding exponents, scaling functions, ratios of amplitudes, etc.; see, e.g., Refs. [7,8], and references therein.

Here and below, by “field theoretic methods” we mean diagrammatic and functional techniques, renormalization theory and renormalization group, composite operators and operator algebras (operator-product or short-distance expansions), instanton calculus, and so on.

Of course, the analogy is far from exact. There is a big difference between the concepts of critical scaling in equilibrium phase transitions and anomalous scaling in turbulence. Formally speaking, in both cases one deals with nontrivial powers of the distance, but in the first case they are divided by the ultraviolet (UV) scale ℓ , while in the second the same role is played by the integral, or infrared (IR) scale L . It was hoped that a close analogy can be achieved if the momentum space for turbulence be confronted with the coordinate space for critical phenomena. This idea was expressed in a phenomenological “dictionary,” where, in particular, the viscous length ℓ (that is, the UV scale of turbulence) was confronted with the correlation length (that is, the IR scale of critical phenomena), while the integral scale L was confronted with the molecular length; see, e.g., Ref. [9]. Hence the idea of “inverse” renormalization group; see Ref. [10] for a recent discussion.

The aforementioned phenomenon of multiscaling was also often opposed to critical scaling, because in the latter “everything is determined by just two exponents η and ν .”

It has usually been stressed that the intermittency is essentially a strongly nonlinear phenomenon, and therefore, the anomalous scaling in turbulence cannot be treated within any kind of perturbation theory. Probably for this reason (and because of a very low quality of some related papers) the field theoretic methods, for many years, have been ignored or taken with a strong skepticism by the turbulent community. The sharpened formulation of the state of the art, given in Ref. [1], is that the results obtained by diagrammatic methods are either wrong or can be derived by much simpler methods (pp. 214–215). Another, and in a sense opposite, point of view was expressed in Ref. [11]: “. . . the reason (that the problem of turbulence is still not solved) lies in the fact that the necessary field theoretic tools have appeared only recently.”

Although the theoretical description of the fluid turbulence on the basis of the stochastic Navier-Stokes (NS) equations remains essentially an open problem, considerable

progress has been achieved in understanding simplified model systems that share some important properties with the real problem: shell models [12], stochastic Burgers equation [13], and passive advection by random “synthetic” velocity fields [14]. Although the shell models, discrete analogs of the NS equation, exhibit pronounced anomalous scaling, it has mostly been studied within numerical simulations. The Burgers equation with random or deterministic initial conditions has been extensively studied analytically, and it exhibits strong intermittency and the energy cascade. The model is interesting in itself and has various applications (e.g., description of the development of singularities in self-gravitating matter), but its relevance for the real hydrodynamical turbulence is far from obvious. In particular, burgulence is a “turbulence without pressure” [11] and, what is more, without the energy conservation (in more than one dimension), while the conservation of energy and the energy exchange between different velocity components are very important features of the genuine fluid turbulence.

Probably the most important progress in the subject, achieved in the last decade of the twentieth century, was related to a simplified model of the fully developed turbulence, the so-called rapid-change model. The model, which dates back to classical studies of Batchelor, Obukhov, Kraichnan, and Kazantsev, describes a scalar or vector quantity (e.g., temperature, concentration of admixture particles, or a weak magnetic field), passively advected by a Gaussian velocity field, decorrelated in time and self-similar in space (the latter property mimics some features of a real turbulent velocity ensemble).

There, for the first time the existence of anomalous scaling was established on the basis of a microscopic model [15], and the corresponding anomalous exponents were derived within controlled approximations [16,17] and regular perturbation schemes [18]. Detailed review of the recent theoretical research on the passive scalar problem and more references can be found in Ref. [14].

It is important to emphasize here that the two alternative (or complementary) analytical approaches to the rapid-change model are both field theoretic. In the “zero-mode approach,” developed in Refs. [16,17] (see also Ref. [14]), nontrivial anomalous exponents are related to the zero modes (unforced solutions) of the closed exact differential equations satisfied by the equal-time correlation functions. From the field theoretic viewpoint, this is a realization of the well-known idea of self-consistent (bootstrap) equations, which involve skeleton diagrams with dressed lines and dropped bare terms (see, e.g., Sec. 4.35 in Ref. [8]). Owing to special features of the rapid-change models (linearity in the passive field and time decorrelation of the advecting field) such equations are exactly given by one-loop approximations, and the resulting equations in the coordinate space are differential (and not integral or integro-differential as in the case of a general field theory). In this sense, the model is “exactly soluble.” Furthermore, in contrast to the case of nonzero correlation time, closed equations are obtained for the *equal-time correlations*, which are Galilean invariant and, therefore, not affected by the so-called “sweeping effects” that would obscure the relevant physical interactions.

In this connection, it should be noted that, due to the time decorrelation, in the rapid-change model there is no problem in relating Eulerian and Lagrangian statistics of the velocity field: they are identical. This allows one to perform very accurate numerical simulations in the Lagrangian frame; see Ref. [20].

From a more physical point of view, zero modes can be interpreted as statistical conservation laws in the dynamics of particle clusters [19]. The concept of statistical conservation laws appears rather general, being also confirmed by numerical simulations of Refs. [21,22], where the passive advection in the two-dimensional NS velocity field [21] and a shell model of a passive scalar [22] were studied. This observation is rather intriguing because in those models no closed equations for equal-time quantities can be derived due to the fact that the advecting velocity has a finite correlation time (for a passive field advected by a velocity with given statistics, closed equations can be derived only for different-time correlation functions, and they involve infinite diagrammatic series).

The second systematic analytical approach to the rapid-change model, proposed in paper [18], is based on the field theoretic renormalization group (RG) and operator-product expansion (OPE).

To avoid possible confusion, it should be explained that in Ref. [18] and subsequent papers, the conventional renormalization group (and not the inverse RG in the spirit of Refs. [9,10]) was employed, which is based on the standard renormalization procedure (elimination of UV divergences). The solution proceeds in two main stages. In the first stage, the multiplicative renormalizability of the corresponding field theoretic model is demonstrated and the differential RG equations for its correlation functions are obtained. The asymptotic behavior of the latter on their UV argument (r/ℓ) for $r \gg \ell$ and any fixed (r/L) is given by IR stable fixed points of those equations. It involves some “scaling functions” of the IR argument (r/L), whose form is not determined by the RG equations. In the second stage, their behavior at $r \ll L$ is found from the OPE within the framework of the general solution of the RG equations. There, the crucial role is played by the critical dimensions of various composite operators, which give rise to an infinite family of independent scaling exponents (and hence to multiscaling).

Of course, both these stages (and thus the phenomenon of multiscaling) have long been known in the RG theory of critical behavior, where the OPE is used in the analysis of the small- (r/L) form of the scaling functions; see, e.g., Refs. [7,8], and references therein. The distinguishing feature, specific to models of turbulence, is the existence of composite operators with *negative* critical dimensions. Such operators are termed “dangerous,” because their contributions to the OPE diverge at $(r/L) \rightarrow 0$. In the models of critical phenomena, nontrivial composite operators always have strictly positive dimensions, so that they only determine corrections [vanishing for $(r/L) \rightarrow 0$] to the leading terms [finite for $(r/L) \rightarrow 0$] in the scaling functions (the leading terms are related to the simplest operator unity with zero critical dimension).

The OPE and the concept of dangerous operators in the

stochastic hydrodynamics were introduced and investigated in detail in Ref. [23]; detailed discussion of the NS case can be found in the review paper [24], the monograph [25], and Chap. 6 of Ref. [8]. Later, the idea of negative dimensions was repeatedly introduced in connection with the anomalous scaling in turbulence [26], models with multifractal behavior [27], and the phenomena related to the Burgers equation [11,28].

The RG analysis of Ref. [18] has shown that dangerous operators are indeed present in the rapid-change model, and that their dimensions can be calculated systematically within a regular perturbation expansions, similar to the famous ε expansion of the critical exponents. Owing to the linearity of the original stochastic equations in the passive field, only finite number of dangerous operators can contribute to any given structure function, which allows one to identify the corresponding anomalous exponent with the critical dimension of an individual composite operator. The actual calculations were performed to the second [18] and third [29] orders in ε (two-loop and three-loop approximations, respectively). Generalizations to the cases of compressible [30,31] and anisotropic [32] velocity ensembles and the vector advected field [33–37] have been obtained.

The two approaches complement each other very well: the zero-mode technique allows for exact (nonperturbative) solutions for the anomalous exponents related to second-order correlation functions [16,38–40] (they are nontrivial for passive vector fields or anisotropic sectors for scalar fields), while the RG approach form the basis for systematic perturbative calculations of the higher-order anomalous exponents [18,29–31]. For the cases of anisotropic velocity ensembles or/and passively advected vector fields, where the calculations become rather involved, all the existing results for higher-order correlation functions were derived only by means of the RG approach and only to the leading order in ε [32–37].

Besides the calculational efficiency, an important advantage of the RG approach is its relative universality: it is not bound to the aforementioned “solubility” of the rapid-change model and can also be applied to the case of finite correlation time or non-Gaussian advecting field [41–43].

It has been usually stressed that intermittency and anomalous scaling in turbulence are signatures of highly nonlinear nature of underlying dynamics. The main lesson that can be learned from the rapid-change model is, probably, that such phenomena can be encountered in a model, which is linear in the passive field, and in which the advecting velocity field is Gaussian and nonintermittent (in contrast to a more realistic case of the stochastic NS equation). What is more, the RG and the OPE approach show that intermittency (at least in the rapid-change model) can be essentially a perturbative phenomenon, in the sense that it is contained completely already in the ordinary (primitive) perturbation theory around a free (Gaussian) approximation. The infinite resummation of the primitive perturbation series, performed by the RG and OPE, gives rise to improved representations of the correlation functions, which reveal anomalous scaling behavior. On the other hand, these representations can be expanded back and

reproduce the original perturbation series, no less and no more.

Existence of exact solutions, regular perturbation schemes, and accurate numerical simulations allows one to discuss, for the example of the rapid-change model and its relatives, the issues that are interesting within the general context of fully developed turbulence: universality and saturation of anomalous exponents, effects of compressibility, anisotropy and pressure, persistence of the large-scale anisotropy and hierarchy of anisotropic contributions, convergence properties and nature of the ε expansions, and so on.

So far, however, aforementioned field theoretic methods have had only limited success when applied to the real fluid turbulence or, better to say, to the stochastic NS equation.

The main problem of the self-consistency approach to the stochastic NS equation is the elimination of the kinematic sweeping effects, which obscure relevant physical interactions and lead to a spurious strong dependence of the correlation functions on the integral scale. This problem was claimed to have been solved using the so-called internal diagrammatic technique [44], but it leads to the violation of the translational invariance. Probably for this reason no attempts have been made to explicitly solve the resulting equations at least in the simplest one-loop approximation.

The standard RG approach to the stochastic NS equation allows one to prove the independence of the inertial-range correlation functions of the viscous scale (the second Kolmogorov hypothesis) and calculate a number of representative constants within a regular ε expansions in a reasonable agreement with experiment; see, e.g., Refs. [8,24,25] for a review and Ref. [45] for the most recent results. The problem of the anomalous scaling, that is, the dependence of the Galilean-invariant correlation functions on the IR scale L , still remains open, probably due to the lack of an appropriate expansion parameter. Dangerous operators in that model are absent in the ε expansions and can appear only at finite values of ε . This means that they can be reliably identified only if their dimensions are derived *exactly* with the aid of Schwinger equations or Galilean symmetry. Due to the non-linear nature of the problem, they enter the corresponding OPE as infinite families whose spectra of dimensions are not bounded from below, and in order to find their dependence on the IR scale L one has to sum up all their contributions. The needed summation of the most singular contributions, related to the powers of the velocity field (their critical dimensions are known exactly), was performed in Ref. [23] with the aid of the so-called infrared perturbation theory for the case of the different-time pair correlation functions. It has revealed their strong dependence on L , which physically can be explained by the aforementioned sweeping effects. This demonstrates that, contrary to what is sometimes claimed, these effects can be properly described within the RG approach, but one should combine the RG and OPE techniques and go beyond the plain ε expansions. Analysis of the L dependence of the Galilean-invariant objects such as the structure functions requires the explicit construction of all dangerous invariant scalar operators, exact calculation of their critical dimensions, and summation of their contributions in the corresponding OPE. This is clearly not a simple

problem and it requires considerable improvement of the present techniques.

As the intermediate step in the investigation of intermittency and anomalous scaling it is important to study simplified models that are, in a number of respects, closer to the real NS turbulence but still allow for analytical treatment. An important step is breaking the artificial assumption of the time decorrelation of the advecting velocity field; see the remarks in Refs. [21,22].

In Refs. [41,42] (see also Ref. [43] for the case of compressible flow) the RG and OPE were applied to the problem of a passive scalar advected by a Gaussian self-similar velocity with finite (and not small) correlation time. The energy spectrum of the velocity in the inertial range has the form $E(k) \propto k^{1-2\varepsilon}$, while the correlation time at the momentum k scales as $k^{-2+\eta}$. It was shown that, depending on the values of the exponents ε and η , the model reveals various types of inertial-range scaling regimes with nontrivial anomalous exponents, which were explicitly derived to the first [41,43] and second [42] orders of the double expansion in ε and η . Earlier, a similar model was proposed and studied in detail (using numerical simulations, in two dimensions) in Ref. [46]. Various aspects of the transport and dispersion of particles in random Gaussian self-similar velocity fields with finite correlation time were also studied in Refs. [47–49].

Another important step toward the NS turbulence is to consider the turbulent advection of passive *vector* fields. The latter can have different physical meaning: magnetic field in the Kazantsev-Kraichnan model of hydromagnetic turbulence in the kinematic approximation; perturbation in the linearized NS equation with prescribed statistics of the background field; density of an impurity with internal degrees of freedom, etc.

Despite the obvious practical significance of these physical situations, the passive vector problem is especially interesting because of the insight it offers into the inertial-range behavior of the NS turbulence. Owing to the coupling between different components of the vector field (both by the dynamical equation and the incompressibility condition) and to the presence of a new stretching term in the dynamical equation, which couples the advected quantity to the gradient of the advecting velocity, the behavior of the passive vector field appears much richer than that of the scalar field: “. . . there is considerably more life in the large-scale transport of vector quantities,” (p. 232 of Ref. [1]). Indeed, passive vector fields reveal anomalous scaling already on the level of the pair correlation function [38]. They also develop interesting large-scale instabilities that can be interpreted as manifestation of the dynamo effect (in kinematic approximation); see, e.g., Refs. [38,50,51]. Other important issues are mixing of composite operators, responsible for the anomalous scaling, and the effects of pressure on the inertial-range behavior, especially in anisotropic sectors.

In the scalar case, the anomalous exponents for all structure functions are given by a single expression which includes n , the order of a function, as a parameter [16–18]. This remains true for the vector models with the stretching term [33,35]. In the special vector model without the stretching term, considered e.g. in [36,37], the number and the form

of the operators entering into the relevant family depend essentially on n , and different structure functions should be studied separately. As a result, no general expression valid for all n exists in the model, and the anomalous exponents are related by finite *families* of composite operators rather than by individual operators [36,37]. In this respect, such a model is closer to the nonlinear NS equation, where the inertial-range behavior of structure functions is believed to be related with the Galilean-invariant operators, which form infinite families that mix heavily in renormalization; see Refs. [24,25].

Another important question that can be addressed for the passive vector model is the effects of nonlocal *pressure* terms on the anomalous scaling, in particular, the consistency of the hierarchical picture for the anisotropic anomalous contributions, known for the pressureless scalar [41,43,52] and magnetic [39,33] cases, with the presence of nonlocal terms in the closed equations for the correlation functions, caused by the pressure contributions [53] (a more detailed treatment is given in Ref. [37]).

The general vector model, introduced in Ref. [35], includes as special cases the kinematic magnetic model, linearized NS equation, and the special model without the stretching term, and thus allows one to control the pressure contribution and quantitatively study its effects on the anomalous scaling. The generalized model also naturally arises within the multiscale technique, as a result of the vertex renormalization [1].

Finally, it should be noted that, as the experience with the passive fields shows, the stochastic NS equation will show anomalous scaling already in its linearized form. Thus the results obtained for the passive vector quantity, with an appropriate statistics of the background field, can be considered as an approximation to the full-fledged NS problem, which, in principle, can be systematically improved by including nonlinear term as the perturbation. It was also argued that, with a proper choice of the random stirring, the passive vector model can reproduce the anomalous exponents of the NS velocity field exactly [54].

The aforementioned works, however, have mostly been confined with the Kraichnan velocity ensemble with zero correlation time.

In the present paper, we consider the model of the passive vector field with the most general form of the nonlinear term, advected by the synthetic velocity ensemble with a finite (and not small) correlation time. The advected velocity field is Gaussian, with the inertial-range spectrum of the form $E(k) \propto k^{1-2\varepsilon}$ and the dispersion law $\omega \propto k^{2-\eta}$, where k is the momentum (wave number). Thus we generalize the general vector model studied in Ref. [35] for the zero-correlated case, to the ensemble of the advecting velocity field employed, e.g., in Refs. [41–43] for the case of passive scalar.

We shall study anomalous scaling, stability of the scaling regimes and analytically derive the anomalous exponents to the first order in $\varepsilon \sim \eta$. This allows us to investigate the universality of the anomalous exponents and the effects of compressibility, pressure, correlation time, and large-scale anisotropy on the inertial-range anomalous scaling. The plan of the paper is as follows.

In Sec. II we give detailed definition of the general vector model and the advecting velocity ensemble and discuss its interesting special cases: the rapid-change and frozen regimes, kinematic dynamo model and linearized NS equation, and so on. We give the field theoretic formulation of the original stochastic problem and present the corresponding diagrammatic technique. In Sec. III we analyze the UV divergences of the model, establish its multiplicative renormalizability, and present the renormalization constants in the one-loop approximation. In Sec. IV we analyze possible scaling regimes of the model, associated with nontrivial and physically acceptable fixed points of the corresponding RG equations. There are seven such regimes, any one of them can be realized depending on the values of the model parameters (ε , η , and others). We discuss the physical meaning of these regimes (e.g., some of them correspond to zero, finite, or infinite correlation time of the advecting field, to the magnetic case, or linearized NS equation) and their regions of stability in the space of the model parameters. In Sec. V we give the scaling representation for a general correlation function and present the general expression for the critical dimension of a composite field (operator). Then we consider the most interesting case of tensor composite operators built only of the passive vector field, which play a crucial role in further discussion of the anomalous scaling. The critical dimension of such an operator with arbitrary number of the fields and vector indices is presented to first order in $\varepsilon \sim \eta$. In Sec. VI we introduce the operator-product expansion and demonstrate its relevance to the issue of inertial-range anomalous scaling. We show that the anomalous exponents in our model can be identified with the critical dimensions of aforementioned composite operators and present the leading terms of the inertial-range behavior for a number of correlation functions. In Sec. VII we show that the anomalous exponents of anisotropic contributions, determined by the critical dimensions of tensor composite fields, obey the hierarchy relations similar to those known for the passive scalar and zero-correlated cases. We also discuss the dependence of the anomalous exponents on the compressibility and pressure, and the effects of these factors on the hierarchy relations. The results obtained are briefly reviewed and discussed in the Conclusion.

II. DESCRIPTION OF THE MODEL: THE FIELD THEORETIC FORMULATION

Here and below, we denote $x \equiv \{t, \mathbf{x}\}$, $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, and d is the (arbitrary) dimensionality of the \mathbf{x} space.

We confine ourselves to the case of transverse (divergence-free) passive vector field $\theta_i(x)$, while the advecting field $\mathbf{v}(x) \equiv \{v_i(x)\}$ may have a longitudinal (potential) component, so that $\partial_i \theta_i = 0$ and $\partial_i v_i \neq 0$. Thus the advection-diffusion equation has the form

$$\partial_t \theta_i + V_i^{(1)} - \mathcal{A}_0 V_i^{(2)} + \partial_i \mathcal{P} = \kappa_0 \partial^2 \theta_i + f_i, \quad (2.1)$$

with the nonlinear terms

$$V_i^{(1)} \equiv \partial_j(v_j \theta_i), \quad V_i^{(2)} \equiv \partial_j(v_i \theta_j) = \theta_j \partial_j v_i. \quad (2.2)$$

Here \mathcal{A}_0 is an arbitrary parameter, $\mathcal{P}(x)$ is the pressure, κ_0 is the diffusivity, ∂^2 is the Laplace operator, and $f_i(x)$ is a Gaussian stirring force with zero mean and correlator

$$\langle f_i(x) f_j(x') \rangle = \delta(t-t') C_{ij}(\mathbf{r}/L), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}'. \quad (2.3)$$

The parameter L is an integral scale related to the stirring and C_{ij} is a dimensionless function finite as $L \rightarrow \infty$. Its precise form is unessential; for generality, it is not assumed to be isotropic. The force $f_i(x)$ maintains the steady state of the system and gives rise to nonzero correlation functions of the field θ . In a more realistic formulation it is replaced by an imposed nonzero mean value $\langle \theta \rangle$, see, e.g., Refs. [39,33].

Nonlinear terms are chosen in the form of total derivatives, so that Eq. (2.1) is the conservation law for θ and, for $\mathcal{A}_0 = 1$, gives the well-known equation for the magnetic field in the hydromagnetic problem. The amplitude factor in front of the first nonlinear term in Eq. (2.1) can be absorbed by rescaling of the velocity field and thus we set it to unity. The third possible structure, $V_i^{(3)} \equiv \partial_i(v_j \theta_j)$, can be absorbed into the pressure term $\partial_i \mathcal{P}$.

Besides the magnetic case ($\mathcal{A}_0 = 1$), model (2.1) includes as special cases the linearized NS equation with prescribed statistics of the background field ($\mathcal{A}_0 = -1$), and the model of passively advected vector impurity ($\mathcal{A}_0 = 0$), which possesses an additional symmetry, $\theta \rightarrow \theta + \text{const}$, and has an intrinsic formal resemblance with the stochastic NS equation; see Ref. [36]. In these examples, the vector field has different physical interpretations: magnetic field, weak perturbation of the prescribed background flow, concentration or density of the impurity particles with an internal structure.

Owing to the transversality condition, the pressure can be expressed as the solution of the Poisson equation,

$$\partial^2 \mathcal{P} = (\mathcal{A}_0 - 1) \partial_i \partial_j (v_j \theta_i). \quad (2.4)$$

For $\mathcal{A}_0 = 1$ (magnetic case) the pressure vanishes. In this case Eq. (2.1) also describes dynamics of the vorticity field advected by a given background velocity field, see, e.g., Ref. [1].

In the real problem, the velocity $\mathbf{v}(x)$ satisfies the NS equation, probably with additional terms that describe the feedback of the advected field $\theta(x)$. We shall begin, however, with a simplified model where the statistics of $\mathbf{v}(x)$ is given: it is a Gaussian field with zero mean and correlation function

$$\begin{aligned} \langle v_i(x) v_j(x') \rangle &= \int \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^d} \{P_{ij}(\mathbf{k}) + \alpha Q_{ij}(\mathbf{k})\} D_v(\omega, k) \\ &\times \exp[-i(t-t') + i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')]. \end{aligned} \quad (2.5)$$

Here $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ and $Q_{ij}(\mathbf{k}) = k_i k_j / k^2$ are the transverse and the longitudinal projectors, respectively, d is the dimensionality of the \mathbf{x} space, $\alpha \geq 0$ is a free parameter ($\alpha = 0$ corresponds to the divergence-free advecting field, $\partial_i v_i = 0$). For the function D_v we choose

$$D_v(\omega, k) = \frac{g_0 u_0 \kappa_0^3 k^{4-d-2\varepsilon-\eta}}{\omega^2 + [u_0 \kappa_0 k^{2-\eta}]^2}. \quad (2.6)$$

For the energy spectrum of the field v we thus obtain $E(k) \simeq k^{d-1} \int d\omega D_v(\omega, k) \simeq g_0 \kappa_0^2 k^{1-2\varepsilon}$. Therefore, the coupling constant $g_0 > 0$ and the exponent ε describe the equal-time velocity correlator or, equivalently, the energy spectrum, while the constant $u_0 > 0$ and the exponent η are related to the frequency $\omega \simeq u_0 \kappa_0 k^{2-\eta}$ characteristic of the mode k . The factor κ_0^3 in the numerator of Eq. (2.6) is explicitly isolated for the convenience later on.

The exponents ε and η are the analogs of the RG expansion parameter $\varepsilon = 4 - d$ in the theory of critical behavior, and we shall use the traditional term “ ε expansion” for the double expansion in the ε - η plane around the origin $\varepsilon = \eta = 0$, with the additional convention that $\eta = O(\varepsilon)$. The IR regularization is provided by the cutoff in integral (2.5) from below at $k \simeq m$, where $m \sim 1/L$ is the reciprocal of the integral scale. Dimensionality considerations show that the coupling constants g_0 , u_0 are related to the characteristic UV momentum scale $\Lambda \sim 1/\ell$ by

$$g_0 \simeq \Lambda^{2\varepsilon}, \quad u_0 \simeq \Lambda^\eta. \quad (2.7)$$

Model (2.5) and (2.6) contains two special cases that possess some interest of their own. In the limit $u_0 \rightarrow \infty$, $g'_0 \equiv g_0/u_0 = \text{const}$, we arrive at the rapid-change model:

$$D_v(\omega, k) \rightarrow g'_0 \kappa_0 k^{-d-\zeta}, \quad \zeta \equiv 2\varepsilon - \eta, \quad (2.8)$$

and the limit $u_0 \rightarrow 0$, $g_0 = \text{const}$ corresponds to the case of a “frozen” velocity field (or “quenched disorder”):

$$D_v(\omega, k) \rightarrow g_0 \kappa_0^2 k^{-d+2-2\varepsilon} \pi \delta(\omega), \quad (2.9)$$

then the velocity correlator (2.5) is independent of the time variable $t-t'$ in the t representation.

The stochastic problem (2.1), (2.3), and (2.5) is equivalent to the field theoretic model of the extended set of three fields $\Phi \equiv \{\theta', \theta, \mathbf{v}\}$ with action functional

$$\begin{aligned} S(\Phi) &= \theta' D_f \theta' / 2 + \theta' [-\partial_t + \kappa_0 \partial^2 \theta - V^{(1)} + \mathcal{A}_0 V^{(2)}] \\ &- \mathbf{v} D_v^{-1} \mathbf{v} / 2, \end{aligned} \quad (2.10)$$

with $V^{(1,2)}$ from Eq. (2.2). This means that statistical averages of random quantities in the original stochastic problem can be represented as functional averages with the weight $\exp S(\Phi)$. The first five terms in Eq. (2.10) represent the so-called Martin-Siggia-Rose action for the stochastic problem (2.1) and (2.3) at fixed \mathbf{v} (see, e.g., Refs. [8,24,25], and references therein), while the last term represents the Gaussian averaging over \mathbf{v} . Here D_f and D_v are the correlation functions (2.3) and (2.5), respectively, $\theta' \equiv \theta'_i(x)$ is an auxiliary transverse vector field, the required integrations over $x = (t, \mathbf{x})$ and summations over the vector indices are implied, for example,

$$\theta' \partial_t \theta \equiv \int dt d\mathbf{x} \theta'_i(t, \mathbf{x}) \partial_t \theta_i(t, \mathbf{x}).$$

TABLE I. Canonical dimensions of the fields and parameters in model (2.10).

F	θ	θ'	\mathbf{v}	ν, ν_0	$m=1/L,$		g_0	u_0	$g, u,$	
					μ, Λ				$\mathcal{A}_0, \mathcal{A}, \alpha$	
d_F^k	0	d	-1	-2	1		2ε	η	0	
d_F^ω	-1/2	1/2	1	1	0		0	0	0	
d_F	-1	$d+1$	1	0	1		2ε	η	0	

The pressure term can be omitted in functional (2.10) owing to the transversality of the auxiliary field:

$$\int dx \theta'_i \partial_i \mathcal{P} = - \int dx \mathcal{P} \partial_i \theta'_i = 0.$$

Of course, this does not mean that the pressure contribution can simply be neglected: the field θ' acts as the transverse projector and selects the transverse part of the expression in the square brackets in Eq. (2.10).

Model (2.10) corresponds to a standard Feynman diagrammatic technique with the triple vertex $\theta'[-V^{(1)} + \mathcal{A}_0 V^{(2)}]$ and bare propagators in the frequency-momentum ($\omega - \mathbf{k}$) representation

$$\begin{aligned} \langle \theta_i \theta'_j \rangle_0 &= \frac{P_{ij}(\mathbf{k})}{(-i\omega + \kappa_0 k^2)}, \\ \langle \theta_i \theta_j \rangle_0 &= \frac{C_{ij}(\mathbf{k})}{(\omega^2 + \kappa_0^2 k^4)}, \quad \langle \theta'_i \theta'_j \rangle_0 = 0, \end{aligned} \quad (2.11)$$

where $C_{ij}(\mathbf{k})$ is the Fourier transform of the function $C_{ij}(\mathbf{r}/L)$ from Eq. (2.3); the bare propagator $\langle v_i v_j \rangle_0$ is given by Eq. (2.5).

III. UV RENORMALIZATION: RG FUNCTIONS AND RG EQUATIONS

The analysis of UV divergences is based on the analysis of canonical dimensions. Dynamical models of type (2.10), in contrast to static models, have two scales, i.e., the canonical dimension of some quantity F (a field or a parameter in the action functional) is described by two numbers, the momentum dimension d_F^k and the frequency dimension d_F^ω ; see, e.g., Refs. [8,24,25]. They are determined so that $[F] \sim [L]^{-d_F^k} [T]^{-d_F^\omega}$, where L is the length scale and T is the time scale. The dimensions are found from the obvious normalization conditions $d_k^k = -d_{\mathbf{x}}^k = 1$, $d_k^\omega = d_{\mathbf{x}}^\omega = 0$, $d_\omega^\omega = d_t^\omega = 0$, $d_\omega^\omega = -d_t^\omega = 1$, and from the requirement that each term of the action functional be dimensionless (with respect to the momentum and frequency dimensions separately). Then, based on d_F^k and d_F^ω , one can introduce the total canonical dimension $d_F = d_F^k + 2d_F^\omega$ (in the free theory, $\partial_i \propto \partial^2$), which plays in the theory of renormalization of dynamical models the same role as the conventional (momentum) dimension does in static problems.

The dimensions for model (2.10) are given in Table I, including the parameters which will be introduced later on.

From the table it follows that the model is logarithmic (the coupling constants g_0, u_0 are dimensionless) at $\varepsilon = \eta = 0$, so that the UV divergences in the correlation functions have the form of the poles in ε, η , and their linear combinations.

The total canonical dimension of an arbitrary one-irreducible correlation function $\Gamma = \langle \Phi \dots \Phi \rangle_{1\text{-ir}}$ is given by the relation $d_\Gamma = d_\Gamma^k + 2d_\Gamma^\omega = d + 2 - N_\Phi d_\Phi$, where $N_\Phi = \{N_\theta, N_{\theta'}, N_{\mathbf{v}}\}$ are the numbers of corresponding fields entering into the function Γ , and the summation over all types of the fields is implied. The total dimension d_Γ is the formal index of the UV divergence. Superficial UV divergences, whose removal requires counterterms, can be present only in those functions Γ for which d_Γ is a non-negative integer.

Analysis of the divergences in our model can be augmented by the following considerations

(i) From the explicit form of the vertex and bare propagators it follows that $N_{\theta'} - N_\theta = 2N_0$ for any one-irreducible correlation function, where $N_0 \geq 0$ is the total number of bare propagators $\langle \theta \theta \rangle_0$ entering into the function. Therefore, the difference $N_{\theta'} - N_\theta$ is an even non-negative integer for any nonvanishing function; cf. Refs. [18,41,43].

(ii) For any model with the Martin-Siggia-Rose-type action, all the one-irreducible functions with $N_{\theta'} = 0$ contain closed contours of retarded propagators $\langle \theta \theta' \rangle_0$ and vanish; see, e.g., Refs. [8,24,25].

(iii) If for some reason a number of external momenta occurs as an overall factor in all the diagrams of a given Green function, the real index of divergence d'_Γ is smaller than d_Γ by the corresponding number (the correlation function requires counterterms only if d'_Γ is a non-negative integer); see, e.g., Refs. [8,24,25]. In model (2.10), the derivative ∂ at the vertex $\theta'[-V^{(1)} + \mathcal{A}_0 V^{(2)}]$ can be moved onto the field θ' using the integration by parts, which decreases the real index of divergence: $d'_\Gamma = d_\Gamma - N_{\theta'}$, and the field θ' enters the counterterms only in the form of the derivative $\partial \theta'$.

From the dimensions in Table I we find $d_\Gamma = d + 2 - N_{\mathbf{v}} + N_\theta - (d+1)N_{\theta'}$ and $d'_\Gamma = (d+2)(1 - N_{\theta'}) + N_\theta - N_{\mathbf{v}}$. Bearing in mind that $N_{\theta'} \geq N_\theta$ we conclude that for any d , superficial divergences can exist only in the one-irreducible functions $\langle \theta' \rangle_{1\text{-ir}}$ with $d_\Gamma = d'_\Gamma = 1$, $\langle \theta' \theta \rangle_{1\text{-ir}}$ with $d_\Gamma = 2$, $d'_\Gamma = 1$, and $\langle \theta' \theta \mathbf{v} \rangle_{1\text{-ir}}$ with $d_\Gamma = 1$, $d'_\Gamma = 0$. The corresponding counterterms necessarily reduce to the forms $\partial \theta'$ (which vanishes identically), $\theta' \partial^2 \theta$, $\theta' V^{(1)}$, and $\theta' V^{(2)}$ with $V^{(1,2)}$ from Eq. (2.2). The structure $\theta' \partial_i \theta$ does not contain a spatial derivative, while $\theta' V^{(3)}$ with $V_i^{(3)} \equiv \partial_i (v_j \theta_j)$ has the form of a total derivative and vanishes after the integration over \mathbf{x} .

We thus conclude that our model (2.10) is multiplicatively renormalizable and the corresponding renormalized action has the form

$$\begin{aligned} S_R(\Phi) &= \theta' D_f \theta' / 2 + \theta' [-\partial_t + \kappa Z_\kappa \partial^2 \theta - Z_1 V^{(1)} + Z_2 \mathcal{A} V^{(2)}] \\ &\quad - \mathbf{v} D_v^{-1} \mathbf{v} / 2. \end{aligned} \quad (3.1)$$

Here and below g, u, κ , and \mathcal{A} (without a subscript) denote the renormalized analogs of the corresponding bare parameters (with the subscript 0). The correlation function D_v in Eq. (3.1) should be expressed in terms of the renormalized parameters and $Z_i = Z_i(g, u, \mathcal{A}, \alpha, \varepsilon, \eta, d)$ are the renormal-

ization constants. The introduction of the counterterms is reproduced by the multiplicative renormalization of the velocity field, $\mathbf{v} \rightarrow Z_1 \mathbf{v}$, and the parameters g_0 , u_0 , κ_0 , and \mathcal{A}_0 in the action functional (2.10):

$$\kappa_0 = \kappa Z_\kappa, \quad u_0 = u \mu^\eta Z_u, \quad g_0 = g \mu^{2\varepsilon} Z_g, \quad \mathcal{A}_0 = Z_A \mathcal{A}. \quad (3.2)$$

Here μ is the reference mass (additional arbitrary parameter of the renormalized theory) and the renormalization constants in Eqs. (3.1) and (3.2) are related as follows:

$$Z_g = Z_1^2 Z_\kappa^{-2}, \quad Z_u = Z_\kappa^{-1}, \quad Z_A = Z_2 Z_1^{-1}. \quad (3.3)$$

The first two relations in Eq. (3.3) result from the absence of the renormalization of the term with D_v in Eq. (3.1). No renormalization of the fields θ, θ' and the parameters $m \sim 1/L$ and α is required, i.e., $Z_\theta = 1$ and so on.

We have calculated all the renormalization constants in the one-loop approximation (first order in g). The resulting expressions are rather cumbersome: they are given by infinite series in the parameter u with the terms containing the poles in $2\varepsilon + s\eta$ with $s=1, 2, \dots$, cf. Refs. [41,43] for the scalar case. For this reason, below we give them only for the special case $\eta=0$ and arbitrary ε . It is important here that the parameter ε alone provides the UV regularization for the theory, so that the constants Z remain finite at $\eta=0$. In the minimal subtraction (MS) scheme they have the form

$$Z_1 = 1 + \frac{g \bar{S}_d}{4d(1+u)^2 \varepsilon} \left\{ \alpha + \frac{\mathcal{A}(1-\mathcal{A})}{(d+2)} - \alpha \frac{(1-\mathcal{A})}{(d+2)} \right\} + O(g^2), \quad (3.4a)$$

$$Z_2 = 1 + \frac{g \bar{S}_d}{4d(1+u)^2 \varepsilon} \left\{ \alpha - \frac{(1-\mathcal{A})}{(d+2)} + \alpha \frac{(1-\mathcal{A})}{\mathcal{A}(d+2)} \right\} + O(g^2), \quad (3.4b)$$

$$Z_\kappa = 1 - \frac{g \bar{S}_d}{4(1+u)\varepsilon} \{ \mathcal{X} + (\alpha - 1) \mathcal{Y} \} + O(g^2), \quad (3.4c)$$

where

$$\mathcal{X} = 1 - \frac{(1-\mathcal{A})^2}{d} - \frac{2}{d(u+1)} + \frac{2(u+2)(1-\mathcal{A})^2}{(u+1)d(d+2)}, \quad (3.5a)$$

$$\mathcal{Y} = \frac{1+\mathcal{A}-\mathcal{A}^2}{d} + \frac{(1-\mathcal{A})^2}{d(d+2)} - \frac{2}{d(u+1)} + \frac{2(1-\mathcal{A})}{d(d+2)(u+1)}. \quad (3.5b)$$

Here and below $\bar{S}_d = S_d / (2\pi)^d$ and $S_d = 2\pi^{d/2} / \Gamma(d/2)$ is the surface area of the unit sphere in d -dimensional space.

The explicit expressions (3.4) illustrate some general properties of the renormalization constants $Z_{1,2}$, valid to all orders in g .

For the magnetic case $\mathcal{A}=1$ we have $Z_1=Z_2$ and, therefore, $Z_A=1$ and the parameter \mathcal{A}_0 is not renormalized: $\mathcal{A}_0 = \mathcal{A}=1$. This is a consequence of the relation $\partial_i [V_i^{(1)} - V_i^{(2)}] = 0$ for vectors (2.2), which expresses the transver-

sality of the vertex $\theta'_i [V_i^{(1)} - V_i^{(2)}]$ with respect to the index of the field θ' . The same property holds for all the diagrams of the one-irreducible function $\langle \theta' \theta v \rangle_{1\text{-ir}}$ and, as a result, for the corresponding counterterm, cf. Refs. [55,56] for the case of the active magnetic field interacting with the NS velocity field.

In the rapid-change limit ($u \rightarrow \infty$, $g/u^2 = \text{const}$) we obtain $Z_1=Z_2=1$ due to the fact that all the diagrams of the function $\langle \theta' \theta v \rangle_{1\text{-ir}}$ contain effectively closed circuits of retarded propagators $\langle \theta \theta' \rangle_0$ and therefore vanish; it is crucial here that the correlation function (2.8) is proportional to the δ function in time representation. On the contrary, in the frozen limit ($u \rightarrow 0$, $g/u = \text{const}$) the constants $Z_{1,2}$ remain non-trivial.

One also obtains $Z_1=Z_2=1$ for $\mathcal{A}=\alpha=0$. In this case, the derivative ∂ at the only vertex $\theta'_i V_i^{(1)} \equiv \theta'_i (\mathbf{v} \partial) \theta_i$ can be moved onto either fields θ and θ' , so that the real index of divergence takes on the form $d'_\Gamma = d_\Gamma - N_\theta - N_{\theta'}$ (we recall that $d'_\Gamma = d_\Gamma - N_{\theta'}$ for general \mathcal{A} and α). This gives $d'_\Gamma = -1$ for $N_\theta = N_{\theta'} = N_v = 1$, so that the function $\langle \theta' \theta v \rangle_{1\text{-ir}}$ is UV finite; cf. Ref. [41] for the scalar case. In other words, the counterterm to the vertex must include two derivatives (one for the field θ and one for θ'), which is forbidden by the dimensionality considerations.

Finally, from Eqs. (3.4) it follows that $Z_{1,2}=1$ for $\mathcal{A}=1$ and $\alpha=0$. We found no general explanation for this relation, but checked that it remains true in the two-loop approximation, so that we can only guarantee that $Z_{1,2} = 1 + O(g^3)$.

Let $W(e_0)$ be some correlation function in the original model (2.10) and $W_R(e, \mu)$ its analog in the renormalized theory with action (3.1). Here e_0 is the complete set of bare parameters and e is the set of their renormalized counterparts. In the following, we shall not be interested in the correlation functions involving the velocity field \mathbf{v} . Then the relation $S(\theta, \theta', Z_2, \mathbf{v}, e_0) = S_R(\theta, \theta', \mathbf{v}, e, \mu)$ for the action functionals yields $W(e_0) = W_R(e, \mu)$ for any correlation function of the fields θ', θ ; the only difference is in the choice of variables and in the form of perturbation theory (in g instead of g_0).

We use $\tilde{\mathcal{D}}_\mu$ to denote the differential operation $\mu \partial_\mu$ for fixed e_0 and operate on both sides of this equation with it. This gives the basic RG equation

$$\mathcal{D}_{RG} W_R(e, \mu) = 0, \quad (3.6)$$

where \mathcal{D}_{RG} is the operation $\tilde{\mathcal{D}}_\mu$ expressed in the renormalized variables

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \beta_g \partial_g + \beta_u \partial_u + \beta_A \partial_A - \gamma_\kappa \mathcal{D}_\kappa. \quad (3.7)$$

In Eq. (3.7), we have written $\mathcal{D}_x \equiv x \partial_x$ for any variable x , the RG functions (the β functions and the anomalous dimensions γ) are defined as

$$\gamma_i \equiv \tilde{\mathcal{D}}_\mu \ln Z_i \quad (3.8)$$

for any renormalization constant Z_i and

$$\begin{aligned}\beta_g &\equiv \bar{\mathcal{D}}_\mu g = 2g[-\varepsilon + \gamma_\kappa - \gamma_1], \\ \beta_u &\equiv \bar{\mathcal{D}}_\mu u = u[-\eta + \gamma_\kappa], \\ \beta_{\mathcal{A}} &\equiv \bar{\mathcal{D}}_\mu \mathcal{A} = \mathcal{A}[\gamma_1 - \gamma_2].\end{aligned}\quad (3.9)$$

The relations between β and γ in Eq. (3.9) result from the definitions and relations (3.3).

For the basis anomalous dimensions from the definitions and expressions (3.4) in the one-loop approximation we obtain

$$\gamma_1 = \frac{-g\bar{S}_d}{2d(1+u)^2} \left\{ \alpha + \frac{\mathcal{A}(1-\mathcal{A})}{(d+2)} - \alpha \frac{(1-\mathcal{A})}{(d+2)} \right\} + O(g^2), \quad (3.10a)$$

$$\gamma_2 = \frac{-g\bar{S}_d}{2d(1+u)^2} \left\{ \alpha - \frac{(1-\mathcal{A})}{(d+2)} + \alpha \frac{(1-\mathcal{A})}{\mathcal{A}(d+2)} \right\} + O(g^2), \quad (3.10b)$$

$$\gamma_\kappa = \frac{g\bar{S}_d}{2(1+u)} \{ \mathcal{X} + (\alpha-1)\mathcal{Y} \} + O(g^2), \quad (3.10c)$$

with \mathcal{X} and \mathcal{Y} from Eq. (3.5). It is also worth noting that the knowledge of the constants Z at $\eta=0$ is in fact sufficient to calculate the β functions (3.9) for all η, ε because the anomalous dimensions (3.10) are independent of η, ε , at least in the one-loop approximation. This fact essentially simplified the two-loop calculation of the anomalous dimensions for the scalar case performed in Ref. [42].

IV. FIXED POINTS AND SCALING REGIMES

It is well known that possible scaling regimes of a renormalizable model are associated with IR attractive fixed points of the corresponding RG equation; see, e.g., Ref. [8]. Roughly speaking, in solving the RG equation all the renormalized coupling constants g_i (that is, all dimensionless parameters of the model) are replaced by the corresponding invariant charges $\bar{g}_i(s)$, where $s \equiv k/\mu$ in the momentum representation or $s \equiv 1/\mu r$ in the coordinate representation. The invariant charges are determined as the solutions of the following Cauchy problem

$$\mathcal{D}_s \bar{g}_i(s) = \beta_i(\{\bar{g}_j(s)\}), \quad \bar{g}_i(1) = g_i \text{ for all } i. \quad (4.1)$$

Here $\bar{g}_i(s)$ is the full set of the invariant charges and $\beta_i \equiv \bar{\mathcal{D}}_\mu g_i$ are the corresponding β functions. In the IR asymptotic range ($s \rightarrow 0$) the invariant charges tend to IR attractive fixed points of system (4.1). The coordinates g_{i*} of the fixed points are found from the equations $\beta_i(\{g_{j*}\}) = 0$ for all i . The type of a fixed point is determined by the matrix $\Omega = \{\Omega_{ij} = \partial\beta_i/\partial g_j\}$: for an IR attractive fixed point the matrix Ω is positive definite, i.e., the real parts of all its eigenvalues are positive.

In our case, the coordinates g_*, u_*, \mathcal{A}_* of the fixed points are found from the equations

$$\beta_g(g_*, u_*, \mathcal{A}_*) = \beta_u(g_*, u_*, \mathcal{A}_*) = \beta_{\mathcal{A}}(g_*, u_*, \mathcal{A}_*) = 0, \quad (4.2)$$

with the β functions given in Eq. (3.9). The coordinates of the fixed points and the elements of the corresponding matrices Ω depend on the remaining free parameters: ε, η, d , and α .¹

Below we list all possible fixed points of system (4.2), giving their coordinates in the one-loop approximation, that is, to first order in ε and η . We shall also present the inequalities that determine the regions (in the space of parameters ε, η, d , and α) where those points are IR attractive. It should also be kept in mind that admissible fixed points should satisfy the relations $g_* \geq 0, u_* \geq 0$, which follow from the physical meaning of these parameters (g is the amplitude of a pair correlation function and u is the ratio of the diffusivity and viscosity coefficients).

First of all, the trivial fixed point

$$g_* = u_* = 0, \quad \mathcal{A}_* \text{ arbitrary} \quad (4.3a)$$

should be mentioned. The corresponding matrix Ω is diagonal with the diagonal elements (eigenvalues)

$$\lambda_1 = 0, \quad \lambda_2 = -2\varepsilon, \quad \lambda_3 = -\eta, \quad (4.3b)$$

so that point (4.3a) is IR attractive for $\varepsilon < 0, \eta < 0$. Since for $g = u = 0$ all the three β functions (3.9) vanish simultaneously, the value of \mathcal{A} at this fixed point remains arbitrary. This degeneracy is reflected in the vanishing of λ_1 .

We shall discuss the physical meaning of point (4.3a) later and now turn to nontrivial fixed points. Their analysis is simplified by the observation that the one-loop function $\beta_{\mathcal{A}}$ factorizes into a part that depends only on \mathcal{A} and a part that depends only on g, u :

$$\beta_{\mathcal{A}} = \frac{g\bar{S}_d}{2d(1+u)^2} (\mathcal{A}^2 - 1)(\mathcal{A} - \alpha) + O(g^2), \quad (4.4)$$

see Eqs. (3.9) and (3.10). Thus all possible values of \mathcal{A}_* are found from the equation $\beta_{\mathcal{A}} = 0$ regardless of the values of the other couplings, provided g is nonzero and g, u are not infinite:

$$\mathcal{A}_* = -1, 1, \alpha. \quad (4.5)$$

Furthermore, the elements $\partial\beta_{\mathcal{A}}/\partial g|_{\mathcal{A}=\mathcal{A}_*} = \partial\beta_{\mathcal{A}}/\partial u|_{\mathcal{A}=\mathcal{A}_*} = 0$ of the matrix Ω vanish for these values of \mathcal{A}_* regardless of the values of g, u . Thus the matrix Ω is block triangular and one of its eigenvalues coincides with the diagonal element $\Omega_{\mathcal{A}} \equiv \partial\beta_{\mathcal{A}}/\partial\mathcal{A}$. Its sign is completely determined by the value of \mathcal{A}_* , so that we can check a necessary condition $\Omega_{\mathcal{A}} > 0$ for a fixed point to be IR attractive regardless of the values of g, u : it is satisfied for $\mathcal{A}_* = -1$ for all α , for $\mathcal{A}_* = 1$ if $\alpha < 1$, and for $\mathcal{A}_* = \alpha$ if $\alpha > 1$. This is easy to under-

¹Formally, α can be treated as the fourth coupling constant. The corresponding β function $\beta_\alpha \equiv \bar{\mathcal{D}}_\mu \alpha$ vanishes identically owing to the fact that α is not renormalized. Therefore, the equation $\beta_\alpha = 0$ gives no additional constraint on the values of the coupling constants at a fixed point.

stand geometrically: $\Omega_{\mathcal{A}} > 0$ for the leftmost and rightmost points and $\Omega_{\mathcal{A}} < 0$ for the point lying in between them.

Therefore, for any α the condition $\Omega_{\mathcal{A}} > 0$ is simultaneously satisfied by two fixed points: $\mathcal{A}_* = -1$ and $\mathcal{A}_* = \max\{1, \alpha\}$. Factorization of function (4.4) allows one to analyze the solution of Eq. (4.1) for the invariant charge $\bar{\mathcal{A}}(s)$ independent of the remaining equations for the other invariant charges: $\bar{\mathcal{A}}(s)$ will be attracted by the leftmost fixed point $\mathcal{A}_* = -1$ if and only if the initial condition $\mathcal{A} = \bar{\mathcal{A}}(1)$ lies to the left of the unstable fixed point, $\mathcal{A} < \min\{1, \alpha\}$, and by the rightmost point $\mathcal{A}_* = \max\{1, \alpha\}$ if and only if $\mathcal{A} > \min\{1, \alpha\}$.

Besides the finite values (4.5), there is one more possibility that formally corresponds to $\mathcal{A}_* = \infty$. More accurately it can be revealed by the change of variables $a \equiv 1/\mathcal{A}$, $y \equiv g\mathcal{A}^2$; then $a_* = 0$ and y_* is finite. It describes the situation when action (2.10) contains the only vertex $\theta' V^{(2)}$. Since in this vertex the derivative can be moved onto either of the fields θ' and \mathbf{v} [see Eq. (2.2)], such a model is multiplicatively renormalizable (the vertex $\theta' V^{(1)}$ is not generated by the renormalization) and $Z_2 = 1$ identically. Thus the result $a_* = 0$ is exact to all orders. However, one can easily check that the eigenvalue of the matrix Ω , equal to the diagonal element $\Omega_a \equiv \partial\beta_a/\partial a$, is always negative, so that this point cannot be IR attractive. We shall not discuss it in what follows.

To conclude the analysis of the equation $\beta_{\mathcal{A}} = 0$, it remains to note that the result $\mathcal{A}_* = 1$ in Eq. (4.5) is exact (a consequence of the relation $Z_1 = Z_2$ for $\mathcal{A} = 1$, see Sec. III), while the other two can have corrections of order $\varepsilon \sim \eta$ and higher. The only exception is the result $\mathcal{A}_* = 0$ for $\alpha = 0$, which is also exact due to the relation $Z_1 = 1$ for $\alpha = 0$.

Substituting values (4.5) into the functions β_g , β_u from Eq. (3.9) and solving the equations $\beta_g = \beta_u = 0$ gives the values of the remaining coordinates g_* and u_* . For each value of \mathcal{A}_* , there are two solutions. For the first of them, $u_* = 0$. This result is exact to all orders of the ε expansion since the function β_u for $u = 0$ vanishes identically, see Eq. (3.9). Substituting the value $u_* = 0$ to β_g and solving the equation $\beta_g = 0$ gives g_* . We shall denote such fixed points by Q , as they correspond to the case of ‘‘quenched disorder,’’ see the comments to Eq. (2.9). For the second variant, g_* and u_* are both nonzero; we shall denote such fixed points by F (‘‘finite’’ correlation time of the velocity field).

Thus we arrive at six nontrivial fixed points $Q^-, Q^+, Q^\alpha, F^-, F^+, F^\alpha$, where the superscripts correspond to the values of \mathcal{A}_* in Eq. (4.5). Below we give the coordinates of these

points in the one-loop approximation, the eigenvalues $\lambda_{1,2,3}$ of the matrix Ω (λ_1 always corresponds to the diagonal element $\partial\beta_{\mathcal{A}}/\partial\mathcal{A}$), and the inequalities that determine the regions where the points are admissible (IR attractive and satisfy $g_* > 0$, $u_* \geq 0$):

$$Q^+: \quad g_* = \frac{2d\varepsilon}{d-1}, \quad u_* = 0, \quad \mathcal{A}_* = 1 \quad (4.6a)$$

with the eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{2\varepsilon(1-\alpha)}{(d-1)(d+2)}, \\ \lambda_2 &= 2\varepsilon, \\ \lambda_3 &= \frac{(d-1-\alpha)\varepsilon - \eta(d-1)}{d-1}, \end{aligned} \quad (4.6b)$$

admissible for

$$\alpha < 1, \quad (d-1-\alpha)\varepsilon > (d-1)\eta. \quad (4.6c)$$

$$Q^-: \quad g_* = \frac{2d(d+2)\varepsilon}{\bar{S}_d(d-2\alpha)(d-1)}, \quad u_* = 0, \quad \mathcal{A}_* = -1, \quad (4.7a)$$

with the eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{2\varepsilon(\alpha+1)}{(d-1)(d-2\alpha)}, \\ \lambda_2 &= 2\varepsilon, \\ \lambda_3 &= \frac{[-2+d-d^2+\alpha(3d-2)]\varepsilon - (2\alpha-d)(d-1)\eta}{(2\alpha-d)(d-1)}, \end{aligned} \quad (4.7b)$$

admissible for

$$\begin{aligned} \varepsilon &> 0, \quad \alpha < d/2, \\ (d-2\alpha)(d-1)\eta &< [d^2-d+2-\alpha(3d-2)]\varepsilon. \end{aligned} \quad (4.7c)$$

$$\begin{aligned} Q^\alpha: \quad g_* &= \frac{2d(d+2)\varepsilon}{d^2-3+\alpha(d+1)(1+\alpha-\alpha^2)}, \\ u_* &= 0, \quad \mathcal{A}_* = \alpha \end{aligned} \quad (4.8a)$$

with the eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{(\alpha^2-1)\varepsilon}{d^2+\alpha(d+1)(1+\alpha-\alpha^2)-3}, \\ \lambda_2 &= 2\varepsilon, \\ \lambda_3 &= \frac{[-3-\alpha+d^2+\alpha^2(1+d)(1-\alpha)]\varepsilon - [-3+d^2+\alpha(1+d)(1+\alpha-\alpha^2)]\eta}{d^2+\alpha(1+d)(1+\alpha-\alpha^2)-3}, \end{aligned} \quad (4.8b)$$

admissible for

$$\varepsilon > 0, \quad \alpha > 1, \quad d^2 - 3 + \alpha(d+1)(1 + \alpha - \alpha^2) > 0,$$

$$\eta < \frac{d^2 - 3 - \alpha + \alpha^2(d+1)(1 - \alpha)}{d^2 - 3 + \alpha(d+1)(1 + \alpha - \alpha^2)} \varepsilon. \quad (4.8c)$$

$$F^+: \quad g_* = \frac{2\alpha d(\eta - 2\varepsilon)^2}{(\alpha + d - 1)^2(\varepsilon - \eta)},$$

$$u_* = \frac{(1 + \alpha - d)\varepsilon + (d - 1)\eta}{(d + \alpha - 1)(\varepsilon - \eta)}, \quad \mathcal{A}_* = 1 \quad (4.9a)$$

with the eigenvalues

$$\lambda_1 = \frac{2(1 - \alpha)(\varepsilon - \eta)}{\alpha(d + 2)},$$

$$\lambda_2 = \frac{W_1 - \sqrt{W_1^2 - W_2}}{2\alpha(2\varepsilon - \eta)},$$

$$\lambda_3 = \frac{W_1 + \sqrt{W_1^2 - W_2}}{2\alpha(2\varepsilon - \eta)}, \quad (4.9b)$$

where

$$W_1 = (2 + 6\alpha - 2d)\varepsilon^2 + 5(d - 1 - \alpha)\varepsilon\eta - 3(d - 1)\eta^2,$$

$$W_2 = 8\alpha(2\varepsilon - \eta)[(1 + \alpha - d)\varepsilon + (d - 1)\eta] \times (2\varepsilon^2 - 3\varepsilon\eta + \eta^2), \quad (4.9c)$$

admissible for

$$\varepsilon > \eta, \quad 2/d < \alpha < (d + 1), \quad 2\varepsilon > \eta, \quad W_1 > 0,$$

$$[2 + \alpha(2 - 3d) + d(d - 1)]\varepsilon < (d - 1)(d - 2\alpha)\eta. \quad (4.10d)$$

$$F^\alpha: \quad g_* = \frac{2\alpha d(2 + d)^2(\eta - 2\varepsilon)^2}{[d^2 - 3 + \alpha^2(1 + d)(1 - \alpha) + \alpha(2d + 3)]^2(\varepsilon - \eta)},$$

$$\mathcal{A}_* = \alpha,$$

$$u_* = \frac{[3 + \alpha - d^2 - \alpha^2(1 + d)(1 - \alpha)]\varepsilon + [d^2 - 3 + \alpha(1 + d)(1 + \alpha - \alpha^2)]\eta}{[d^2 - 3 + \alpha^2(1 + d)(1 - \alpha) + \alpha(2d + 3)](\varepsilon - \eta)} \quad (4.11a)$$

$$\varepsilon > 0, \quad \eta > 0, \quad \alpha < 1, \quad \varepsilon > \eta > \frac{d - \alpha - 1}{d - 1} \varepsilon,$$

$$(2 + 6\alpha - 2d)\varepsilon^2 + 5(d - 1 - \alpha)\varepsilon\eta - 3(d - 1)\eta^2 > 0. \quad (4.9d)$$

$$F^-: \quad g_* = \frac{2d(2 + d)(\alpha d - 2)(\eta - 2\varepsilon)^2}{(1 - \alpha + d)^2(d - 2)^2(\varepsilon - \eta)},$$

$$u_* = \frac{[2 + \alpha(2 - 3d) - d + d^2]\varepsilon + (2\alpha - d)(d - 1)\eta}{(\alpha - d - 1)(d - 2)(\varepsilon - \eta)}, \quad \mathcal{A}_* = -1 \quad (4.10a)$$

with the eigenvalues

$$\lambda_1 = \frac{2(1 + \alpha)(\varepsilon - \eta)}{-2 + \alpha d},$$

$$\lambda_2 = \frac{W_1 - \sqrt{W_1^2 - W_2}}{2(-2 + \alpha d)(2\varepsilon - \eta)},$$

$$\lambda_3 = \frac{W_1 + \sqrt{W_1^2 - W_2}}{2(-2 + \alpha d)(2\varepsilon - \eta)}, \quad (4.10b)$$

where

$$W_1 = 2[-6 + d - d^2 + \alpha(-2 + 5d)]\varepsilon^2 + 5[2 + \alpha(2 - 3d) - d + d^2]\varepsilon\eta + 3(2\alpha - d)(-1 + d)\eta^2,$$

$$W_2 = 8(-2 + \alpha d)(2\varepsilon - \eta)^2(\varepsilon - \eta) \times \{[-2 + d - d^2 + \alpha(-2 + 3d)]\varepsilon - (2\alpha - d)(-1 + d)\eta\}, \quad (4.10c)$$

admissible for

with the eigenvalues

$$\begin{aligned}\lambda_1 &= \frac{(\alpha^2 - 1)(\varepsilon - \eta)}{\alpha(2 + d)}, \\ \lambda_2 &= \frac{W_1 - \sqrt{W_1^2 - W_2}}{2\alpha(2 + d)(2\varepsilon - \eta)}, \\ \lambda_3 &= \frac{W_1 + \sqrt{W_1^2 - W_2}}{2\alpha(2 + d)(2\varepsilon - \eta)},\end{aligned}\quad (4.11b)$$

where

$$\begin{aligned}W_1 &= 2[3 - d^2 - \alpha^2(1 + d) + \alpha^3(1 + d) + \alpha(5 + 2d)]\varepsilon^2 \\ &\quad - 5[3 + \alpha - d^2 - \alpha^2(1 + d) + \alpha^3(1 + d)]\varepsilon\eta \\ &\quad + 3[3 - d^2 - \alpha(1 + d)(1 + \alpha - \alpha^2)]\eta^2, \\ W_2 &= 8\alpha(2 + d)(2\varepsilon - \eta)^2(\varepsilon - \eta) \\ &\quad \times [(3 + \alpha - d^2 - \alpha^2(1 + d) \\ &\quad \times (1 - \alpha)]\varepsilon + [-3 + d^2 + \alpha(1 + d)(1 + \alpha - \alpha^2)]\eta],\end{aligned}\quad (4.11c)$$

admissible for

$$\begin{aligned}\alpha > 1, \quad \varepsilon > \eta, \quad 2\varepsilon > \eta, \quad W_1 > 0, \\ [d^2 - 3 + \alpha^2(1 + d)(1 - \alpha) + \alpha(2d + 3)] > 0, \\ [3 + \alpha - d^2 - \alpha^2(1 + d)(1 - \alpha)]\varepsilon \\ + [d^2 - 3 + \alpha(1 + d)(1 + \alpha - \alpha^2)]\eta > 0.\end{aligned}\quad (4.11d)$$

However, the above list is not exhaustive: besides the fixed points with noninfinite values of g_* and u_* , there are points for which these parameters tend to infinity. They can be revealed by the change of variables $x \equiv g/u$, $w \equiv 1/u$. The corresponding β functions are obtained by the chain rule:

$$\beta_x \equiv \tilde{\mathcal{D}}_\mu x = (1/u)\beta_g - (g/u^2)\beta_u, \quad \beta_w \equiv \tilde{\mathcal{D}}_\mu w = -(1/u^2)\beta_u, \quad (4.12)$$

with $\beta_{g,u}$ from Eq. (3.9) and the anomalous dimensions (3.10) expressed in the variables x , w . Solving the equations $\beta_x = \beta_w = \beta_{\mathcal{A}} = 0$ gives three fixed points with finite values of x_* , w_* , which simply express the points $F^{\pm, \alpha}$ in the new variables. Besides them, there are two fixed points with $w_* = 0$, left out in the analysis performed in terms of g , u . It is clear from Eq. (2.8) that the choice $w = 0$ corresponds to the rapid-change limit of our model. The dimension γ_κ from Eq. (3.10) remains finite for $w = 0$, while γ_1 and γ_2 vanish. In fact, they vanish to all orders of the perturbation theory in $x^\alpha g$ owing to the exact relation $Z_1 = Z_2 = 1$ that holds in the limit (2.8); see the discussion in Sec. III. As a result, the function $\beta_{\mathcal{A}}$ vanishes identically for $w = 0$ and the coordinate \mathcal{A}_* remains arbitrary at such fixed points; see the remark below Eq. (4.3a).

There is a trivial fixed point

$$x_* = w_* = 0, \quad \mathcal{A}_* \text{ arbitrary} \quad (4.13a)$$

with the eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = \eta, \quad \lambda_3 = \eta - 2\varepsilon, \quad (4.13b)$$

and a nontrivial fixed point, which we shall denote by R^A in what follows,

$$\begin{aligned}R^A: x_* &= \frac{2d(d+2)(2\varepsilon - \eta)}{\mathcal{A}(1 + \alpha)d + d^2 + \alpha(3 + d) - \mathcal{A}^2(-1 + \alpha + \alpha d) - 3}, \\ w_* &= 0, \quad \mathcal{A}_* \text{ arbitrary}\end{aligned}\quad (4.14a)$$

with the eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = -2(\varepsilon - \eta), \quad \lambda_3 = 2\varepsilon - \eta, \quad (4.14b)$$

admissible for

$$\begin{aligned}\eta > \varepsilon > \eta/2, \\ \mathcal{A}(1 + \alpha)d + d^2 + \alpha(3 + d) - \mathcal{A}^2(-1 + \alpha + \alpha d) - 3 > 0.\end{aligned}\quad (4.14c)$$

The notation R^A implies that this point corresponds to the ‘‘rapid-change’’ regime and that $\mathcal{A}_* = \mathcal{A}$ remains a free parameter.

Triviality of points (4.3a) and (4.13a) implies the absence of anomalous scaling; they correspond to diffusive-type regimes, for which the convection [that is, the nonlinearity in Eq. (2.1)] can be treated within ordinary perturbation theory and the standard methods of the homogenization theory apply. More detailed discussion of such fixed points (in particular, the difference between the ‘‘quenched’’ and rapid-change trivial fixed points) can be found in Ref. [41] for the example of the passive scalar field, advected by the velocity ensemble (2.5).

On the contrary, the nontrivial fixed points $Q^{\pm, \alpha}$, $F^{\pm, \alpha}$, and R^A describe nondiffusive asymptotic regimes, in which the competition of the diffusive and convective terms in Eq. (2.1) produces anomalous scaling behavior. The corresponding anomalous exponents will be presented in the following section, and now we shall discuss the interplay between the possible scaling regimes. Seven nontrivial IR attractive fixed points correspond to seven possible scaling regimes; only one of them can be realized when the values of all parameters α , ε , η , d , and \mathcal{A} are given, regardless of the values of the amplitudes g and u (see below). In this sense, the asymptotic behavior in our model is universal, and (as we shall see) the anomalous exponents depend only on the values of the exponents ε and η in the velocity correlation function and on the parameters α (for all regimes) and \mathcal{A} (for R^A), but they do not depend on the coupling constants g and u .

Indeed, let us fix the value of \mathcal{A} . Then the solution of the RG equation (3.6) can be attracted by either of the following three sets: by the set $\{F^-, Q^-, R^A\}$ if $\mathcal{A} < \min\{1, \alpha\}$, by the set $\{F^+, Q^+, R^A\}$ if $\alpha < 1$ and $\mathcal{A} > \alpha$, and by the set

$\{F^\alpha, Q^\alpha, R^A\}$ if $\alpha > 1$ and $A > 1$. [In all these cases, the value of \mathcal{A}_* for the fixed point R^A in Eq. (4.14a) simply equals to \mathcal{A}].

For any of these three situations, only one fixed point in the list $\{F, Q, R\}$ can be IR attractive for given ε and η . Indeed, the analysis of the inequalities that determine the admissibility regions for these points shows that these regions adjoin each other without overlaps or gaps. The common boundary of the admissibility regions for the points F and R is $\varepsilon = \eta$ [one of the admissibility conditions for R^A is $\varepsilon < \eta$, see Eq. (4.14c), while one of the admissibility conditions for any of the points $F^{\pm, \alpha}$ is $\varepsilon > \eta$, see Eqs. (4.9d), (4.10d), and (4.11d)]. The common boundary of the admissibility regions for the points F and Q is $(d-1-\alpha)\varepsilon = (d-1)\eta$ for the pair Q^+, F^+ [see Eqs. (4.6c) and (4.9d)],

$$[d^2 - 3 + \alpha(d+1)(1 + \alpha - \alpha^2)]\eta = [d^2 - 3 - \alpha + \alpha^2(d+1)(1 - \alpha)]\varepsilon$$

for the pair Q^-, F^- [see Eqs. (4.7c) and (4.10d)] and $[d^2 - 3 + \alpha^2(1+d)(1-\alpha) + \alpha(2d+3)]\varepsilon = 0$ for the pair F^α, Q^α [see Eqs. (4.8c) and (4.11d)].

Therefore, for any given set of parameters $\alpha, \varepsilon, \eta, d$, and \mathcal{A} , the RG trajectory can be attracted by the only one possible nontrivial fixed point, regardless of the values of the parameters g, u [that is, the initial data for the Cauchy problem (4.1)]. If the trajectory is attracted by a trivial point, the model will show diffusive-type behavior. Finally, if there is no IR attractive fixed point for the given set of parameters, no definitive conclusion can be made about the asymptotic behavior of the model within the framework of the ε expansion. In particular, if the trajectory comes to a region where the parameters g, u are negative, one can think that the steady state of our system becomes unstable (by analogy with the RG theory of critical phenomena, where such behavior is usually interpreted as a first-order phase transition).

The admissibility regions in the ε - η plane are shown in Fig. 1 for a number of values of the parameters α and d . In the one-loop approximation, their boundaries are always given by straight rays starting at the origin ε - η . Some of them, however, can be affected by the higher-order corrections, so that the gaps or overlaps of different regions can appear in the two-loop approximation (Fig. 2).

It is interesting to note that the scaling regimes that arise as solutions of the RG equations for the general model, in several cases correspond to interesting physical situations (Fig. 3). In particular, the regimes governed by the points $Q^{\pm, \alpha}$ correspond to the case of time-independent (or frozen) velocity field, while $F^{\pm, \alpha}$ correspond to the rapid-change limit of the general model (2.5).

To avoid possible misunderstandings we emphasize that the limits $u_0 \rightarrow 0$ or $g_0 \rightarrow \infty, u_0 \rightarrow \infty$ are not supposed to be performed in the original correlation function (2.5); the parameters g_0, u_0 (and hence g, u) are fixed at some finite values. The behavior specific to models (2.8) and (2.9) arises asymptotically as a result of the solution of the RG equations, when the RG flow approaches the corresponding fixed point. This shows that in the regimes governed by the points

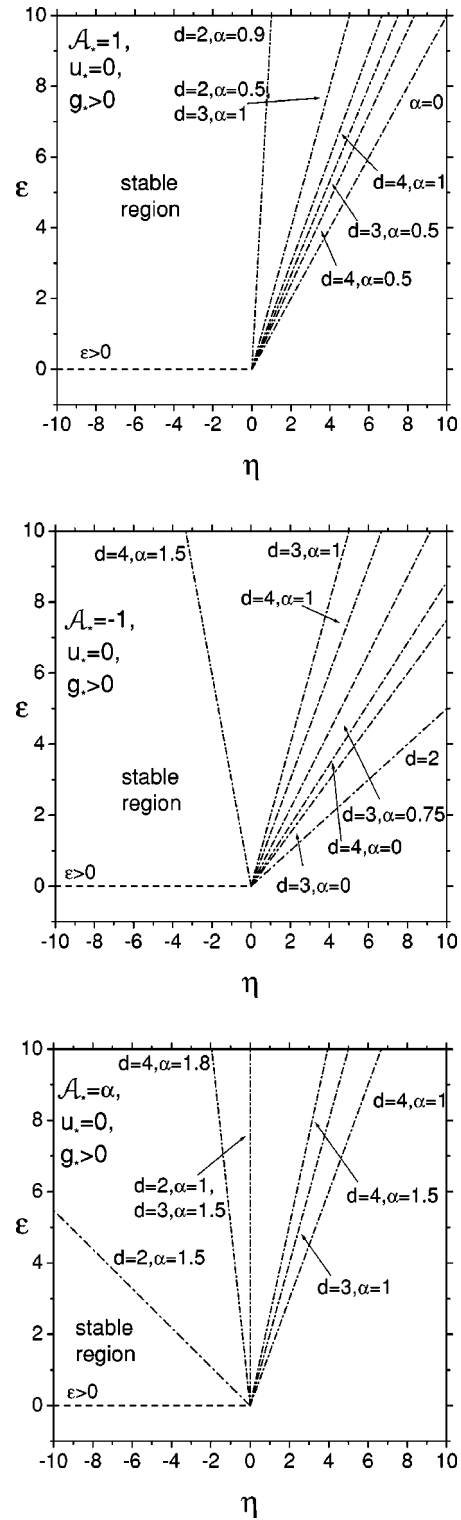


FIG. 1. Regions of infrared stability of the three fixed points (4.6a), (4.7a), and (4.8a) corresponding to quenched disorder for a representative set of values of the space dimension d and the parameter α of the relative strength of the longitudinal part in the correlation function of the velocity field. The region of stability for any indicated values of d and α lies between the dashed ray ($\varepsilon = 0, \eta \leq 0$) and the correspondingly marked dash-dotted ray in the upper half of the η - ε plane. Note different scales on the coordinate axes.

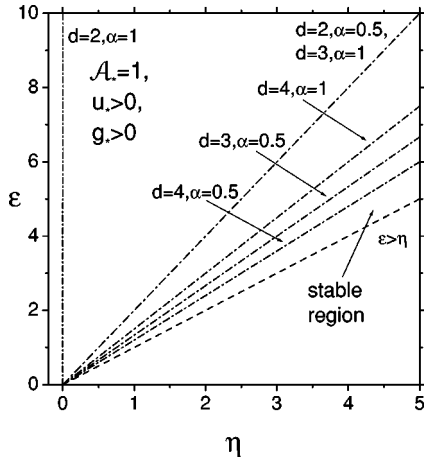


FIG. 2. Regions of infrared stability of the fixed point (4.9a) with a finite correlation time in the asymptotic regime. Basins of attraction are shown for a representative set of values of the space dimension d and the parameter α of the relative contribution of the longitudinal part in the correlation function of the velocity field. The region of stability for any indicated values of d and α lies between the dashed ray ($\varepsilon = \eta$, $\eta \geq 0$) and the correspondingly marked dash-dotted ray in the upper half of the η, ε plane. Note different scales on the coordinate axes.

$Q^{\pm, \alpha}$, the temporal fluctuations of the velocity field are asymptotically irrelevant in determining the inertial-range behavior of the passive field, which is then completely determined by the equal-time velocity statistics. In the regimes governed by R^A , spatial and temporal fluctuations are both relevant, but the effective correlation time of the passive field becomes so large under renormalization that the correlation time of the velocity can be completely neglected (Fig. 4). The inertial-range behavior of the passive field is determined solely by the $\omega=0$ mode of the velocity field; this is the case of the rapid-change model. In particular, this means that the coordinates of the fixed points and the anomalous exponents in such regimes must depend on the only exponent $\zeta \equiv 2\varepsilon - \eta$ or ε that survives in the limit in question, and coincide with the corresponding dimensions obtained directly for models (2.8) or (2.9). This is indeed the case, as one can see from the explicit expressions given above and in the following section.

As regards the value of the amplitude factor \mathcal{A} in Eq. (2.1), it can remain an arbitrary parameter (for R^A), or can be attracted by one of the fixed points $\mathcal{A}_* = \pm 1$, or α (for Q or F); see Eq. (4.5). Again, these possibilities correspond to physical situations interesting as such. The case $\mathcal{A}_* = 1$ corresponds to the behavior characteristic of the magnetic model, where the pressure vanishes due to relation (2.4) and Eq. (2.1) becomes local. Therefore, the infrared behavior of the nonlocal model can be described by the same fixed point (or universality class) as that of the local (magnetic) model; the nonlocal pressure term does not affect the asymptotic properties of the passive field. The general model indeed becomes a turbulence without pressure. The case $\mathcal{A}_* = -1$ corresponds to the linearized NS equation with a given statistics of the background field (we recall, however, that this value of \mathcal{A}_* can be affected by the higher-order corrections

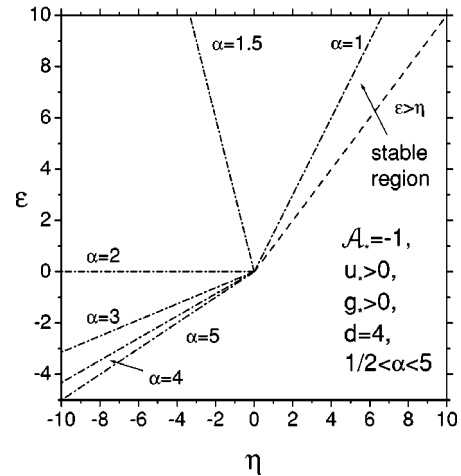
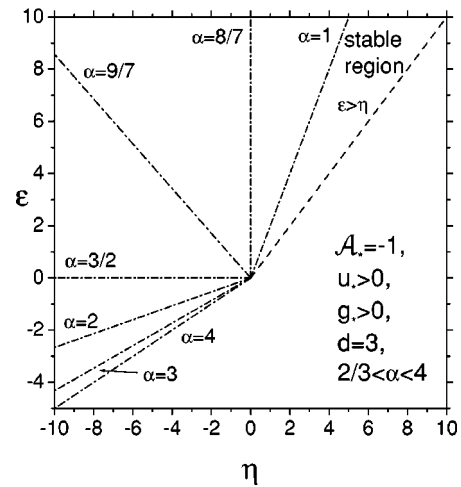
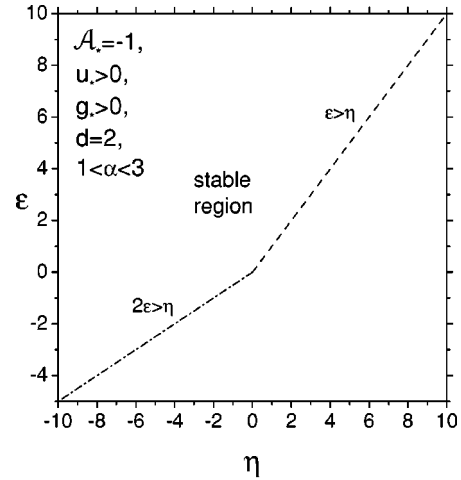


FIG. 3. Regions of infrared stability of the fixed point (4.10a) with a finite correlation time in the asymptotic regime. Basins of attraction of the three fixed points are shown for a representative set of the longitudinal parameter α in space dimensions two, three, and four. The region of stability for any indicated value of α lies from the dashed ray ($\varepsilon = \eta$, $\eta \geq 0$) to the left up to the correspondingly marked dash-dotted ray. Note different scales on the coordinate axes. Contrary to the quenched-disorder case, negative values of the correlation falloff parameter ε are (formally) allowed.

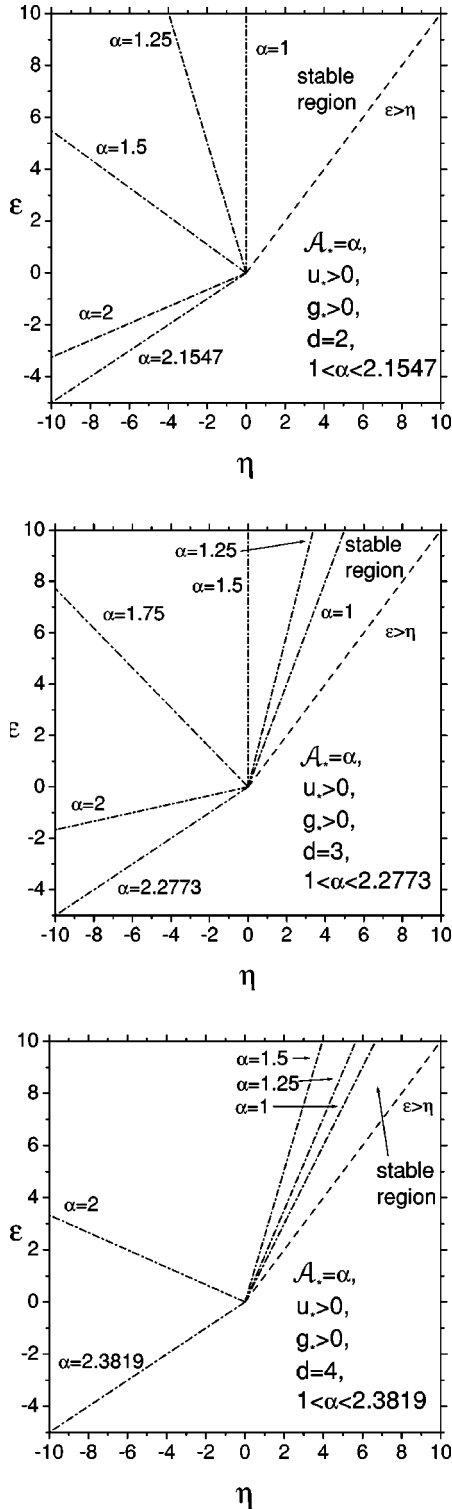


FIG. 4. Regions of infrared stability of the fixed point (4.11a) with a finite correlation time in the asymptotic regime. Basins of attraction of the three fixed points are shown for a representative set of the longitudinal parameter α in space dimensions two, three, and four. The region of stability for any indicated value of α lies from the dashed ray ($\varepsilon = \eta$, $\eta \geq 0$) to the left up to the correspondingly marked dash-dotted ray. Note different scales on the coordinate axes. Negative values of ε appear here as well as for the fixed point (4.11a).

in ε and η). Finally, the case $\mathcal{A}_* = 0$, $\alpha = 0$ corresponds the model, which (in its rapid-change variant) was introduced independently in a number of studies as an example of a linear system with pressure [37,53], a model with nontrivial mixing of composite operators [36,37] or a model which, with a proper choice of the forcing, can reproduce the anomalous exponents of the NS velocity field [54].

It is worth noting that for $\alpha \neq 0$, the model with $\mathcal{A}_0 = 0$ is not renormalizable, as follows from the analysis given in Sec. III. That is, the second nonlinear term $V_i^{(2)}$ will be generated by the renormalization procedure, even if it was absent in the original equation (2.1). Of course, this fact does not mean that the model with $\mathcal{A}_0 = 0$ and $\alpha \neq 0$ is inconsistent; it rather means that its IR behavior is described by one of the fixed point with $\mathcal{A}_* \neq 0$.

V. CRITICAL SCALING: CRITICAL DIMENSIONS OF COMPOSITE OPERATORS

Consider for definiteness some equal-time two-point quantity $F(r)$ that depends on a single distance parameter r , for example, the pair correlation function of the primary fields θ, θ' or some composite operators. We assume that $F(r)$ is multiplicatively renormalizable, i.e., $F = Z_F F^R$ with certain renormalization constant Z_F . Then the function $F^R(r)$ satisfies the RG equation of the form $[\mathcal{D}_{RG} + \gamma_F] F(r) = 0$ with the operator \mathcal{D}_{RG} from Eq. (3.7) and $\gamma_F \equiv \bar{D}_\mu \ln Z_F$, cf. Eq. (3.8). The functions F and F^R are equally suitable for studying the asymptotic behavior: the difference is in the normalization, choice of parameters (bare or renormalized), and the form of the perturbation theory (in g_0 or in g). The solution of the RG equation can be written in terms of the invariant variables introduced in Eq. (4.1). The analysis shows that in the IR asymptotic region, defined by the inequality $\Lambda r \gg 1$ with Λ from Eq. (2.7) and any fixed mr with $m = 1/L$ from Eq. (2.3), the invariant charges approach one of the IR attractive fixed points (the choice of the appropriate fixed point is discussed in the preceding section), and, as a result, the function $F(r)$ takes on the self-similar form

$$F(r) \approx \kappa_0^{d_F^\omega} \Lambda^{d_F} (\Lambda r)^{-\Delta_F} \xi(mr), \quad (5.1)$$

where d_F^ω and d_F are the frequency and total canonical dimensions of F , respectively (see Sec. III and Table I), and ξ is some function whose explicit form is not determined by the RG equation itself. The critical dimension Δ_F of the quantity F is given by the expression

$$\Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^* = d_F - \gamma_\kappa^* d_F^\omega + \gamma_F^*, \quad (5.2)$$

where γ_F^* denotes the value of the anomalous dimension γ_F at the fixed point in question, and $\Delta_\omega = 2 - \gamma_\kappa^*$ with γ_κ from Eq. (3.7) is the critical dimension of frequency.

Each nontrivial fixed point $Q^{\pm, \alpha}$, $F^{\pm, \alpha}$, and R^A from Sec. IV corresponds to the scaling representation of the form (5.1) with its own set of critical dimensions Δ_F for all quantities F and Δ_ω . In general, these dimensions are infinite series in ε

and η . For the rapid-change regime (that is, for the point R^A) they depend on the only exponent $\zeta \equiv 2\varepsilon - \eta$ from (2.8), while for the regimes with quenched disorder (that is, for the points $Q^{\pm, \alpha}$) they depend on the only exponent ε that survives in limit (2.9). For the fixed point R^A , the equation $\beta_x = 0$ determines the values of $\gamma_\kappa^* = \zeta$ and $\Delta_\omega = 2 - \zeta$ exactly, that is, without corrections of order ζ^2 and higher. This follows from the explicit expressions (3.9) and (4.12) and the vanishing of the anomalous dimensions $\gamma_{1,2}$; see the discus-

sion below Eq. (4.12). For the fixed points $F^{\pm, \alpha}$ with $u_* \neq 0$ the equation $\beta_u = 0$ leads to the exact result $\gamma_\kappa^* = \eta$; see Eq. (3.9). For the fixed point Q^α with $\mathcal{A}_* = \alpha = 0$, the exact result $\gamma_\kappa^* = \varepsilon$ follows from the equation $\beta_g = 0$ and vanishing of γ_1 ; see the discussion in Sec. III. For the other regimes, the first-order expressions for γ_κ^* are directly obtained by substituting the coordinates of the fixed points (4.6a), (4.7a), and (4.8a) into the one-loop expression (3.10c) for γ_κ . The results can be summarized as follows:

$$\gamma_\kappa^* = \begin{cases} \zeta \equiv 2\varepsilon - \eta \text{ (exact)} & \text{for } R^A \\ \eta \text{ (exact)} & \text{for } F^{\pm, \alpha} \\ \varepsilon \frac{(d-1-\alpha)}{(d-1)} & \text{for } Q^+ \\ \varepsilon \frac{(d^2-d-3d\alpha+2\alpha+2)}{(d-1)(d-2\alpha)} & \text{for } Q^- \\ \varepsilon \frac{(d^2-d+2)}{d(d-1)} & \text{for } Q^- \text{ and } \alpha=0 \\ \varepsilon \frac{(d^2-d\alpha^3+d\alpha^2-\alpha^3+\alpha^2-\alpha-3)}{(d^2-d\alpha^3+d\alpha^2+d\alpha-\alpha^3+\alpha^2+\alpha-3)} & \text{for } Q^\alpha \\ \varepsilon \text{ (exact)} & \text{for } Q^\alpha \text{ and } \alpha=0. \end{cases} \quad (5.3)$$

Then the dimensions Δ_F , e.g., of the primary fields θ, θ' are obtained from Eq. (5.2) and the data from Table I,

$$\Delta_\theta = -1 + \gamma_\kappa^*/2, \quad \Delta_{\theta'} = d - \gamma_\kappa^*/2, \quad (5.4)$$

and the dimensions of their correlation functions are given by simple sums over the fields entering into the function.

In the following, the crucial role will be played by the critical dimensions $\Delta[n, l]$ associated with the irreducible tensor composite fields (“local composite operators” in the field theoretic terminology) built solely of the fields θ at a single space time point $x = \{t, \mathbf{x}\}$. They have the form

$$F[n, l] \equiv \theta_{i_1}(x) \cdots \theta_{i_l}(x) [\theta_i(x) \theta_i(x)]^p + \cdots, \quad (5.5)$$

where $l \leq n$ is the number of the free vector indices and $n = l + 2p$ is the total number of the fields θ entering into the operator; the vector indices and the argument x of the symbol $F[n, l]$ are omitted. The dots stand for the appropriate subtractions involving the Kronecker δ symbols, which ensure that the resulting expressions are traceless with respect to contraction of any given pair of indices, for example, $\theta_i \theta_j - \delta_{ij} \theta_k \theta_k / d$, $\theta_i \theta_j \theta_k - (\delta_{ij} \theta_k + \delta_{ik} \theta_j + \delta_{jk} \theta_i) \theta^2 / (d+2)$ and so on. We also note that the numbers n and l are even or odd simultaneously.

Owing to the coincidence of the arguments, additional UV divergences arise in the correlation functions involving such operators. They are eliminated by means of the additional renormalization procedure, which gives rise to new [inde-

pendent of $Z_{1,2,\kappa}$ from Eq. (3.1)] renormalization constants. The analysis similar to that given in Refs. [41,43] for the scalar and in [33] for the magnetic model shows that, in the case at hand, these constants can be calculated in the model without forcing [the bare propagator $\langle \theta_i \theta_j \rangle_0$ from Eq. (2.11) does not enter into the corresponding Feynman diagrams]. Then operators (5.5) appear multiplicatively renormalizable: $F[n, l] = Z[n, l] F^R[n, l]$.

We have calculated all the constants $Z[n, l]$ in the one-loop approximation (first order in g) in the MS scheme for the special case $\eta=0$ and arbitrary ε , which is sufficient to find the corresponding anomalous dimensions $\gamma[n, l] = \bar{D}_\mu \ln Z[n, l]$; cf. the discussion above Eq. (3.4). We omit the calculation (which is very similar to that performed in Refs. [41,43,33] for the scalar and magnetic cases) and give only the final result:

$$Z[n, l] = 1 + \frac{g \bar{S}_d}{8\varepsilon d(d+2)(u+1)} [\mathcal{A}^2 Q_1 + \alpha Q_2], \quad (5.6)$$

where

$$Q_1 = n(d+n)(d-1) - l(l+d-2)(d+1),$$

$$Q_2 = n(dn+n-d)(d-1) - l(l+d-2). \quad (5.7)$$

For the anomalous dimension we thus obtain

$$\gamma[n,l] = \frac{-g\bar{S}_d}{4d(d+2)(u+1)} [\mathcal{A}^2 Q_1 + \alpha Q_2]. \quad (5.8)$$

According to Eq. (5.2), the critical dimensions of operators (5.5) are given by

$$\Delta[n,l] = n\Delta_\theta + \gamma^*[n,l], \quad (5.9)$$

with Δ_θ from Eq. (5.4) and $\gamma^*[n,l]$ is the value of the

anomalous dimension (5.8) at one of the nontrivial fixed points (4.6a),(4.7a),(4.8a),(4.9a),(4.10a),(4.11a),(4.14a). Note that owing to the renormalization, the critical dimensions of operators (5.5) differ from the naive sum of the dimensions of the fields θ that constitute the operator. The exception is provided by the case $\mathcal{A}_* = \alpha = 0$, when $\Delta[n,l] = n\Delta_\theta$ exactly (the proof is similar to that given, e.g., in Ref. [18] for the scalar case).

The results for the anomalous dimensions $\gamma^*[n,l]$ can be summarized as follows:

$$\gamma^*[n,l] = -(2\varepsilon - \eta) [\mathcal{A}_*^2 Q_1 + \alpha Q_2] / 2 \times \begin{cases} \frac{1}{(d+2)(d-1+\alpha)} & \text{for } F^+ \\ \frac{1}{(d-2)(d+1-\alpha)} & \text{for } F^- \\ \frac{1}{(d^2 - d\alpha^3 + d\alpha^2 + 2d\alpha - \alpha^3 + \alpha^2 + 3\alpha - 3)} & \text{for } F^\alpha \\ \frac{1}{(d^2 - 3)} & \text{for } F^\alpha \text{ and } \alpha = 0 \end{cases} \quad (5.10)$$

for the regimes with finite correlation time,

$$\gamma^*[n,l] = -\varepsilon [\mathcal{A}_*^2 Q_1 + \alpha Q_2] / 2 \times \begin{cases} \frac{1}{(d+2)(d-1)} & \text{for } Q^+ \\ \frac{1}{(d-1)(d-2\alpha)} & \text{for } Q^- \\ \frac{1}{(d^2 - d\alpha^3 + d\alpha^2 + d\alpha - \alpha^3 + \alpha^2 + \alpha - 3)} & \text{for } Q^\alpha \\ \frac{1}{(d^2 - 3)} & \text{for } Q^\alpha \text{ and } \alpha = 0 \end{cases} \quad (5.11)$$

for the regimes with quenched disorder, and

$$\gamma^*[n,l] = -(2\varepsilon - \eta) [\mathcal{A}^2 Q_1 + \alpha Q_2] / 2 \times \begin{cases} \frac{1}{(d^2 + \mathcal{A}^2 - \alpha\mathcal{A}^2 - \alpha d\mathcal{A}^2 + \alpha d\mathcal{A} + d\mathcal{A} + d\alpha + 3\alpha - 3)} & \text{for } R^{\mathcal{A}}, \\ \frac{1}{(d^2 + \mathcal{A}^2 + d\mathcal{A} - 3)} & \text{for } R^{\mathcal{A}} \text{ and } \alpha = 0 \end{cases} \quad (5.12)$$

for the regimes with zero correlation time. We recall that \mathcal{A}_* takes on the values 1, -1 , and α for the fixed points F and Q labeled by the superscripts $+$, $-$, and α , respectively, while for the rapid-change regime $\mathcal{A}_* = \mathcal{A}$ remains an arbitrary parameter. We also note that dimensions (5.11) depend on the only exponent ε that survives in limit (2.9), while dimensions (5.12) depend on the only exponent $\zeta = 2\varepsilon - \eta$ that survives in limit (2.8). Exponents (5.10) also depend only on ζ , which seems to be an artifact of the first-order approximation. We also note that exponents (5.12) were de-

rived earlier directly for the rapid-change model (2.8) for some special cases: $\mathcal{A} = 1$ and $\alpha = 0$ in Ref. [33], $\mathcal{A} = 1$ and arbitrary $\alpha \geq 0$ in Ref. [34] and arbitrary \mathcal{A} and $\alpha = 0$ in Ref. [35].

From Eqs. (5.3), (5.4), and (5.9)–(5.12) it follows that

$$\Delta[n,l] = -n + O(\varepsilon). \quad (5.13)$$

Thus for all nontrivial fixed points, at least for small values

of $\varepsilon \sim \eta$, one has $\Delta[n,l] < \Delta[k,j]$ if $n > k$ regardless of the relation between l and j . For a fixed value of n one has

$$\Delta[n,l] > \Delta[nj] \quad \text{and} \quad \gamma[n,l] > \gamma[nj] \quad \text{if} \quad l > j, \quad (5.14)$$

as one can easily see from the original expression (5.8) for the anomalous dimension $\gamma[n,l]$, properties of polynomials (5.7) and the fact that the combination $g/(u+1) = x/(w+1)$ that enters into Eq. (5.8) is positive definite for all nontrivial fixed points. [We recall that the parameters x, w were introduced above Eq. (4.12) for the proper description of the rapid-change regimes.] Thus for fixed n , the dimension $\Delta[n,l]$ decreases monotonically with l and reaches its minimum for the minimal possible value of l , that is, $l=0$ if n is even and $l=1$ if n is odd.

The hierarchy relations (5.13) and (5.14) will be important in the discussion of the inertial-range behavior of various correlation functions, in particular, in the issue of the large-scale anisotropy persistence; see Sec. VI. Similar inequalities were established earlier for the case of the passive scalar field, advected by the velocity ensemble (2.5) and (2.6) in Refs. [41,43], and for the magnetic field advected by the Kraichnan ensemble (2.8) in Ref. [39,33].

It also follows from Eq. (5.13) that the critical dimensions $\Delta[n,l]$ are *negative* and that the spectrum of their dimensions is not bounded from below; these properties, typical of the models of turbulence, will also be important in the following.

If the random force f is introduced in Eq. (2.1), none of formulas (5.6)–(5.12) change, because the bare correlator $\langle \theta\theta \rangle_0$ from Eq. (2.11), which becomes nonzero, does not enter the relevant diagrams. However, operators (5.5) are no longer renormalized multiplicatively: the operator $F[k,j]$ can admix in renormalization to the operator $F[n,l]$ if and only if, $k < l$. In general, $j \neq l$, but if correlator (2.3) is isotropic, only operators with $j=l$ can mix in renormalization. The admixture of operators with $k > l$ is impossible due to the absence of appropriate diagrams; this is a consequence of the linearity of the original equation (2.1) in θ and f . The admixture of operators with $k=n$ and $j \neq l$ is also impossible, because the corresponding diagrams do not involve the correlator $\langle \theta\theta \rangle_0$ and therefore do not “feel” the violation of the rotational symmetry caused by function (2.3).

As a result of the mixing, the operator $F[n,l]$ becomes a finite sum of contributions with definite critical dimensions $\Delta[n,l]$ and $\Delta[k,j]$ with $k < n$ and, in general, all possible values of j allowed for a given k . However, due to relation (5.13), the leading term is still given by the original contribution with the dimension $\Delta[n,l]$, while the new contributions with $k < n$ give only corrections that vanish in the IR range $\Lambda r \gg 1$ in expressions such as (5.1). In what follows, we shall be interested only in the leading terms of and thus we can ignore the mixing and treat the operator $F[n,l]$ as if it has the definite critical dimension $\Delta[n,l]$.

VI. OPERATOR PRODUCT EXPANSION AND THE ANOMALOUS SCALING

Representation (5.1) for any scaling function $\xi(mr)$ describes the behavior of the correlation function $F(r)$ for $\Lambda r \gg 1$ and any fixed value of mr . The inertial range corresponds to the additional condition that $mr \ll 1$. The form of the function $\xi(mr)$ is not determined by the RG equations themselves; in the theory of critical phenomena, its behavior for $mr \rightarrow 0$ is studied using the well-known Wilson operator-product expansion (OPE); see, e.g., Ref. [7]. This technique is also applicable in the theory of turbulence; see, e.g., Refs. [23–25].

According to the OPE, the equal-time product $F_1(x)F_2(x')$ of two renormalized composite operators at $\mathbf{x} \equiv (\mathbf{x} + \mathbf{x}')/2 = \text{const}$ and $\mathbf{r} \equiv \mathbf{x} - \mathbf{x}' \rightarrow 0$ can be represented in the form

$$F_1(x)F_2(x') = \sum_F C_F(\mathbf{r})F(t, \mathbf{x}), \quad (6.1)$$

where the functions C_F are the Wilson coefficients regular in m^2 and F are, in general, all possible renormalized local composite operators allowed by symmetry; more precisely, the operators entering into the OPE are those which appear in the corresponding Taylor expansions, and also all possible operators that admix to them in renormalization. If these operators have additional vector indices, they are contracted with the corresponding indices of the coefficients C_F .

Without loss of generality it can be assumed that the expansion in Eq. (6.1) is made in the operators with definite critical dimensions Δ_F . The renormalized correlation function $\langle F_1(x)F_2(x') \rangle$ is obtained by averaging Eq. (6.1) with the weight $\exp S_R$ with S_R from Eq. (3.1), the quantities $\langle F \rangle$ appear on the right-hand side. Their asymptotic behavior for $m \rightarrow 0$ is found from the corresponding RG equations and has the form $\langle F \rangle \propto m^{\Delta_F}$.

From the operator-product expansion (6.1) we therefore find the following expression for the scaling function $\xi(mr)$ in the representation (5.1) for the correlation function $\langle F_1(x)F_2(x') \rangle$:

$$\xi(mr) = \sum_F A_F(mr)^{\Delta_F}, \quad (6.2)$$

where the coefficients $A_F(mr)$ are regular in $(mr)^2$.

The quantities of interest are, in particular, the equal-time pair correlation functions of the composite operators (5.5). For these, representation (5.1) is valid with the dimensions $d_F^{\omega} = -(n+k)/2$, $d_F = -(n+k)$ and $\Delta_F = \Delta[n,l] + \Delta[k,j]$ with $\Delta[n,l]$ from Eq. (5.9):

$$\begin{aligned} & \langle F[n,l](t, \mathbf{x})F[k,j](t, \mathbf{x}') \rangle \\ & = (\kappa_0)^{-(n+k)/2} \Lambda^{-(n+k)} (\Lambda r)^{-\Delta[n,l] - \Delta[k,j]} \xi(mr), \end{aligned} \quad (6.3)$$

where $\Lambda r \gg 1$ and $\xi(mr)$ is the corresponding scaling function.

As already said above, the operators entering into the OPE are those which appear in the corresponding Taylor expansions, and also all possible operators that admix to them in renormalization. The leading term of the Taylor expansion for function (6.3) is given by the n th rank tensor $F[n+k, l+j]$ from Eq. (5.5). Its decomposition in irreducible tensors gives rise to the operators $F[n+k, p]$ with all possible values of $p \leq l+j$; the admixture of junior operators (see the end of Sec. V) gives rise to all the monomials $F[s, p]$ with $s < n+k$ and all possible p allowed for a given s . Hence, the asymptotic expression for the structure function $\xi(mr)$ for $mr \ll 1$ has the form

$$\xi(mr) = \sum_{s=0}^{n+k} \sum_{p=p_s}^s [A_{sp}(mr)^{\Delta[s,p]} + \dots], \quad (6.4)$$

with the dimensions $\Delta[k, p]$ from Eq. (5.9). Here and below p_s denotes the minimal possible value of p for given s , i.e., $p_s = 0$ for k even and $p_s = 1$ for k odd; A_{sp} are some numerical coefficients dependent on the parameters such as ε , d , and so on. The dots in Eq. (6.4) stand for the contributions which arise from the composite operators that, in addition to the field θ , involve the other fields θ' , \mathbf{v} and/or derivatives ∂, ∂_i .

The leading term of expression (6.4) for $mr \ll 1$ is determined, obviously, by the minimal possible dimension Δ_F that appear on its right-hand side, provided this minimal dimension exists. In our model, there are infinitely many operators with negative critical dimensions, and the spectrum of their dimensions is not bounded from below. (It is possible to show on general grounds that if a model involves one negative dimension it necessarily involves infinitely many negative dimensions with unbounded spectrum.) If all these operators appeared on the right-hand side of representation (6.4), we would have to sum up their contributions in order to find the asymptotic behavior at $mr \rightarrow 0$. This problem is indeed encountered for the stochastic NS equation [23], and is discussed in Refs. [25,24] in detail.

In our model, however, there is no such problem, at least for small ε . The contributions of the operators $F[s, p]$ with $s > n+k$ (which would be more important) do not appear in Eq. (6.4), because they are absent in the Taylor expansion of correlator (6.3) and do not admix in renormalization to the terms of the Taylor expansion; see Sec. V. As already noted there, this is a manifestation of the linearity of the original equation (2.1) in θ and f . What is more, one can show that for any operator F that appear in the OPE [and not only operators (5.5) built solely of the fields θ] the number of the fields θ cannot exceed the total number of the fields θ on the left-hand side; therefore their dimensions cannot appear in Eq. (6.4). It then follows that the leading term in Eq. (6.4) is determined by an operator built solely of the fields θ and containing the maximal possible number of the fields, that is, $n+k$. The operators containing less than $n+k$ fields θ give only corrections, as follows from the hierarchy relation (5.13). The operators involving the fields θ' , \mathbf{v} and/or derivatives also give only corrections, because the canonical dimensions of these additional factors are positive (see Table

I) and thus increase the total canonical dimension of the operator in comparison with the corresponding operator built solely of the fields θ . Furthermore, from the hierarchy relation (5.14) it follows that, for the fixed number of the fields θ , the minimum of the dimension is achieved for the minimal number of vector indices, that is, for the scalar operator $F[n+k, 0]$ if the sum $n+k$ is even and the vector operator $F[n+k, 1]$ if the sum $n+k$ is odd.

We therefore conclude that the leading term of the small- mr behavior of the scaling function (6.4) has the form $\xi \sim (mr)^{\Delta[n+k, l_{n+k}]}$. Substituting this expression into Eq. (6.3) gives the desired leading term of the correlation function of two operators (5.5) in the inertial range ($\Lambda r \gg 1$, $mr \ll 1$):

$$\begin{aligned} & \langle F[n, l](t, \mathbf{x}) F[k, j](t, \mathbf{x}') \rangle \\ & \sim (\kappa_0)^{-(n+k)/2} \Lambda^{-(n+k)} (\Lambda r)^{-\Delta[n, l] - \Delta[k, j]} \\ & \times (mr)^{\Delta[n+k, l_{n+k}]}, \end{aligned} \quad (6.5)$$

with the dimensions $\Delta[n, l]$ from Eq. (5.9).

VII. ANOMALOUS SCALING IN ANISOTROPIC SECTORS: HIERARCHY OF ANISOTROPIC CONTRIBUTIONS

Without loss of generality, it can always be assumed that expansion (6.1) is made in irreducible traceless tensor composite operators. Then averaging Eq. (6.1) with the weight $\exp S_R$ automatically produces the decomposition of the correlation function in irreducible representations of the rotation group $SO(d)$, similar to that employed, e.g., in Refs. [57–62] for description of the NS turbulence. This becomes especially clear if the left-hand side of Eq. (6.1) involves only scalar quantities and the anisotropy, introduced by correlator (2.3), is uniaxial, that is, specified by a single-constant unit vector \mathbf{n} . Then the mean value $\langle F \rangle$ of a l th rank tensor operator F on the right-hand side of Eq. (6.1) is an irreducible traceless l th rank tensor built only of the vector \mathbf{n} and the Kronecker delta symbols. Its vector indices are contracted with the indices of the corresponding Wilson coefficient $C_F(\mathbf{r})$, which gives rise to the Legendre (or Gegenbauer for arbitrary d) polynomial of order l . In general, decomposition in (hyper)spherical harmonics (see, e.g., Ref. [64] and references therein) or its analogs for tensor quantities (see, e.g., Ref. [65] and the references therein) will be encountered.

The rank l of the operator can be viewed as the measure of anisotropy of the corresponding contribution in expansion (6.4). If the forcing is isotropic, that is, the function $C(\mathbf{r})$ in correlator (2.3) depends only on $r = |\mathbf{r}|$, only scalar operators with $l=0$ have nonvanishing mean values, and only their dimensions appear on the right-hand side of Eq. (6.4). In general, tensor operators with $l \neq 0$ also contribute to Eq. (6.4). Owing to relations (5.14), the leading term of the asymptotic behavior at $mr \rightarrow 0$ is still given by the scalar operator with $l=0$ (it has the minimal dimension among the operators with a fixed number of the fields). We thus conclude that the leading term is given by the same expression (6.5) for both the isotropic and anisotropic forcing, while

anisotropic contributions with $l > 0$ give only subleading terms (corrections). What is more, relations (5.14) show that these contributions reveal a kind of *hierarchy* related to the degree of anisotropy: the higher is the rank of the operator, the less important is its contribution to the inertial-range behavior.

For the first time, the hierarchy relations for anisotropic contributions were derived in Ref. [39] for the magnetic field, passively advected by Kraichnan's velocity ensemble (2.8), and in Ref. [41] for the scalar field, advected by the Gaussian velocity field specified by correlator (2.6) (in both cases with general d and $\alpha = 0$). In the first of these papers, anomalous exponents were found exactly for the pair correlation function, while in the second the exponents were derived only in the one-loop approximation, but for all the higher-order correlation functions. Later these results were reproduced in Ref. [40] for the magnetic field (only for $d = 3$, but also including helical contributions) and in Ref. [52] for the scalar field advected by Kraichnan's ensemble. Generalization to the higher-order correlation functions of the magnetic field was given in Ref. [33] (for $\alpha = 0$), while generalizations to the case of general α were obtained in Ref. [34] (magnetic field and Kraichnan's ensemble) and in Ref. [43] [scalar field and ensemble (2.6)]. Generalization to the general vector model (2.1) and Kraichnan's ensemble was given in Ref. [35].

So far, analytical results of such kind have been obtained only for passive fields, advected by the synthetic Gaussian velocity ensembles. However, numerical simulations and real experiments show that the picture outlined above appears rather general, being observed by the passive scalar field advected by the two-dimensional NS field in the inverse energy cascade [63] and by the NS velocity field itself [59–62]. These observations justify and make more precise old phenomenological ideas about the isotropization of the inertial-range turbulence in the presence of a large-scale anisotropy. Nevertheless, the anisotropy survives in the inertial range and reveals itself in *odd* correlation functions, in disagreement with what was expected on the basis of the cascade ideas. We shall return to this important issue in the Conclusion, and now let us briefly discuss the influence of compressibility on the hierarchy of the anomalous exponents.

Effects of the compressibility on the anomalous scaling in anisotropic sectors were studied earlier for the scalar [31] and magnetic [34] fields advected by Kraichnan's ensemble (2.8) and for the scalar advected by the velocity ensemble (2.6) with finite correlation time [43]; see also Ref. [66] for a summary. In all those cases the conclusion was the same: the hierarchy expressed by relation (5.14), which can be rewritten as

$$\partial\Delta[n,p]/\partial p > 0 \quad (7.1)$$

remains valid for all values of the compressibility parameter $0 \leq \alpha < +\infty$, but it always becomes less pronounced as α grows,

$$\partial^2\Delta[n,p]/\partial p\partial\alpha < 0. \quad (7.2)$$

This means, in particular, that the anisotropic corrections in Eq. (6.4) become closer to each other and to the leading term as α grows. Thus the compressibility enhances the penetration of the large-scale anisotropy into the inertial range. This penetration is even more manifest for the odd-order ratios of the correlation functions: the skewness factor grows for $mr \rightarrow 0$, provided α is large enough, while the growth of the hyperskewness factor and other higher-order ratios becomes much faster than for the incompressible case; see the discussion in Refs. [31,43,34,66].

No such definite conclusions can be drawn for the general vector model. The straightforward analysis of the explicit expressions (5.10)–(5.12) shows that the derivative $\partial^2\Delta[n,p]/\partial p\partial\alpha$ is negative and, therefore, the behavior described above takes place only in two regimes described by the fixed points F^+ (always) and R^A (only if the relation

$$\begin{aligned} -3 + \mathcal{A}d + d^2 - \mathcal{A}^3d(d+1) + \mathcal{A}^4(d+1)^2 \\ - \mathcal{A}^2(d^2 + 4d + 2) > 0, \end{aligned} \quad (7.3)$$

which is independent of α , is satisfied). For all the other cases, one finds $\partial^2\Delta[n,p]/\partial p\partial\alpha > 0$ and the behavior is opposite: compressibility suppresses the penetration of the large-scale anisotropy into the inertial range, anisotropic contributions become farther from one another and from the isotropic term.

It is tempting to attribute this “inverse behavior” to the combined influence of the compressibility and pressure. Indeed, the “normal” (scalar-like) behavior takes place for the magnetic regime F^+ , for which $\mathcal{A}_* = \mathcal{A}_0 = 1$ and the pressure term (2.4) vanishes. For the rapid-change regime R^A and reasonable values of d , relation (7.3) is satisfied only in the restricted area around the point $\mathcal{A}_* = 1$ (including $\mathcal{A}_* = 1$, in agreement with the analysis of Ref. [34]), where the pressure effects are relatively small. However, the magnetic regime in a frozen velocity field Q^+ demonstrates the inverse behavior.

The dependence on the parameter \mathcal{A} , which controls the pressure effects, is essentially different from the dependence on α : for the regimes with nonzero (finite or infinite) correlation time, the value of the corresponding invariant variable can take only three discrete values $\mathcal{A}_* = -1, 1, \alpha$. The value of \mathcal{A}_* which is realized for a given regime depends on α but not on \mathcal{A} ; see Eq. (4.5). The case $\mathcal{A}_* = 1$ corresponds to the magnetic (pressureless) equations; then the general model indeed becomes a turbulence without pressure, despite the presence of the nonlocal pressure term in the original stochastic equation. For the zero-correlated regime R^A , the anomalous exponents retain a continuous dependence on \mathcal{A} ; for the incompressible case ($\alpha = 0$) it was discussed in Ref. [35]. The value of $\mathcal{A} = 1$, where the pressure effects disappear, is not distinguished at all; the derivative $\partial^2\Delta[n,p]/\partial p\partial\mathcal{A}$ at $\mathcal{A} = 1$ is positive for almost all parameters (namely, for $d > -1.3\alpha + 1.5$, the approximate relation obtained numerically), but is negative definite, e.g., for $\mathcal{A} = 0$.

VIII. CONCLUSION

We have studied a model of a divergence-free (transverse) vector quantity passively advected by a random Gaussian velocity field with finite (and not small) correlation time. The model is described by an advection-diffusion equation with a random large-scale stirring force, nonlocal pressure term, and the most general form of the inertial nonlinearity. The correlation function of the advecting field mimics some properties of the real inertial-range turbulence: the energy spectrum has the form $E(k) \propto k^{1-2\varepsilon}$, while the correlation time scales as $k^{-2+\eta}$. An advantage of the model is the possibility to control the pressure contribution and thus study its effects on the inertial-range behavior. Another reason to study the general case is the possibility to describe in a uniform way several special cases interesting on their own: the kinematic magnetic model, linearized NS equation, and the special model without the stretching term, which possesses additional symmetry and has a close formal resemblance with the nonlinear NS case.

We have shown that the system exhibits various types of inertial-range asymptotic behavior, characterized by nontrivial anomalous exponents; the latter are analytically calculated to first order in $\varepsilon \sim \eta$, including the anisotropic sectors.

The key points of our analysis are the existence of a field theoretic formulation of the original stochastic problem (Sec. II), multiplicative renormalizability of the corresponding field theory (Sec. III), existence of nontrivial IR attractive fixed points of the corresponding RG equations in the physical region of the parameters (Sec. IV), and the possibility to identify the anomalous exponents with the critical dimensions of certain composite operators (Secs. V and VI). This allows one to construct a systematic perturbation expansion for the exponents; the practical calculations have been performed to the first nontrivial order in $\varepsilon \sim \eta$ (one-loop approximation).

Existence of explicit one-loop expression allows one to discuss the stability scaling regimes and the universality of the corresponding exponents, that is, their (in)dependence on the pressure, anisotropy, compressibility, forcing, and so on, or, more technically, on the exponents ε, η and the amplitudes \mathcal{A}_0, g_0, u_0 , and α in the stochastic equation (2.1) and correlator (2.6) of the advecting velocity. Although the behavior of the vector model is much richer than that of its scalar counterpart, the general picture appears essentially the same: the exponents are universal in the sense that they depend on the exponents ε and η , but do not depend on the amplitudes in Eq. (2.6) and forcing (2.3); the exponents related to anisotropic contributions show a hierarchy related to the degree of anisotropy (more anisotropic contributions are less important); this hierarchy holds for all scaling regimes, regardless of the values of the compressibility parameter α from Eq. (2.5) and the pressure parameter \mathcal{A} from Eq. (2.1). Consider these points in more detail.

Scaling regimes and universality classes. Infrared asymptotic behavior of our model is completely described by seven different scaling regimes, or universality classes, each corresponding to a set of anomalous exponents. For the given set of the parameters ε, η , and α , only one of these

regimes can be realized, irrespective of the values of the amplitudes \mathcal{A}_0, g_0, u_0 . Three regimes correspond to finite correlation time of the advecting velocity field, and three regimes correspond to infinite correlation time (or time-independent velocity). Each of these two sets involve the magnetic (pressureless) case, linearized NS equation, and the model with α -dependent effective amplitude in front of the stretching term; for $\alpha=0$ the latter gives the special model with no stretching term, whose rapid-change version was studied, e.g., in Refs. [37,53,54]. The scaling exponents for these regimes depend on the exponents ε, η and the amplitude α , but they are independent of the values of the amplitudes \mathcal{A}_0, g_0, u_0 . The remaining seventh regime corresponds to the rapid-change velocity (zero correlation time), the corresponding exponents depend also on \mathcal{A}_0 .

To avoid possible confusion we stress that the behavior specific to the aforementioned classes, e.g., magnetic model with infinite correlation time, arises automatically when the RG flow approaches the fixed point which is IR attractive for the given choice of parameters $\varepsilon, \eta, \alpha$; the frozen limit (2.9) or the substitution $\mathcal{A}_0=1$ are not performed in the original model and the parameters \mathcal{A}_0, g_0, u_0 are fixed at such finite values. In particular, this means that the anomalous exponents in those regimes are independent of the correlation time (more precisely, the ratio of the correlation time of the velocity field and the turnover time for the scalar field, measured by the parameter u_0 ; see the discussion in Ref. [41]). In this sense, one can speak about the universality of the anomalous exponents in our model.

Independence of the forcing: Zero-mode picture. As we have seen, the critical dimensions of all composite operators (5.9), and therefore the corresponding anomalous exponents (including anisotropic sectors), are independent of the forcing, specified by correlator (2.3). In particular, this means that they remain unchanged, when the stirring force in Eq. (2.1) is replaced by the imposed mean constant field, such as in Refs. [33,39]. The role of the forcing is to maintain the steady state of the system and thus to provide nonzero amplitudes for the power-like terms with those universal exponents.

This behavior is already well known for the passive scalar fields [41,43] advected by velocity (2.6) or vector fields, advected by the zero-correlated velocity [33,39].

In the language of the RG (which is equally applicable to the case of a zero or finite correlation time) this is explained as follows: the stirring force or the mean field do not enter into the diagrams that determine the renormalization of operators (5.9), so that their dimensions are independent of the forcing. Similar diagrams determine the contributions of those operators into the operator-product expansions (6.1), which are nontrivial even for the unforced model. The difference is that for the unforced model, mean values of the operators vanish, and they give no contribution to the right-hand sides of representations such as Eq. (6.4). For the isotropic correlator (2.3), scalar operators acquire nonzero mean values and contribute to the right-hand side of Eq. (6.4), while for the anisotropic correlator or the imposed mean field, the mean values of irreducible tensor operators also

become nonzero and their contributions are “activated” in representations (6.4).

For the case of zero correlation time, when the equal-time correlations functions satisfy exact closed differential equations, the above picture it is easily understood in the language of the zero-mode approach [14]: forcing terms do not affect the corresponding differential operators; thus the anomalous exponents, determined by the zero modes (solutions of homogeneous unforced equations) also are independent of the forcing. On the contrary, the *amplitudes* are determined by the matching of the inertial-range zero-mode solution with the forced large-scale solutions, which is only possible in the presence of the forcing terms.

The exact resemblance in the behavior of the rapid-change models and the finite-correlated cases suggests that for the latter, the concept of zero modes (and thus of statistical conservation laws) is also applicable, although the corresponding equations are not differential and involve infinite diagrammatic series.

Hierarchy of anisotropic contributions. In the presence of the large-scale anisotropy (that is, the anisotropy introduced at scales of order L by the forcing), correlation functions of the model can be decomposed in irreducible representations of the d -dimensional rotation group $SO(d)$. Such a decomposition naturally arises from the corresponding OPE, provided it is made in irreducible traceless tensor composite operators; the rank l of a tensor operator can be used to label the terms of the $SO(d)$ expansion and can be viewed as the measure of anisotropy of the corresponding term (“sector”). Thus each anisotropic sector is characterized by its own set of scaling exponents, the leading term is given by the l th rank composite operator with minimal critical dimension.

Explicit expressions for these dimensions were obtained to the first order in ε and η . They reveal an hierarchy related to the degree of anisotropy: the higher is the rank of the operator (the more anisotropic is the contribution), the larger is the corresponding dimension, and thus the less important is its contribution to the inertial-range behavior.

This hierarchy, expressed by relations (5.14) or (7.1), holds for all nontrivial scaling regimes of our model, all values of the parameters α , \mathcal{A} , d , and so on. It is similar to the hierarchy relations derived earlier for the passive scalar [41,43,52] and magnetic fields [33,39,40] advected by the Gaussian velocity ensembles.

In particular, this means that the overall leading term is given by the exponent from the isotropic sector, and it is therefore the same for the isotropic and anisotropic forcing. It also should be stressed that the independence of the scaling behavior in different sectors is a direct consequence of the linearity of our model, independence of the exponents on the random force, and the $SO(d)$ symmetry of the unforced model. On the contrary, the *hierarchy* of the exponents follows from the explicit expressions, obtained only by practical calculation.

According to the Kolmogorov-Obukhov theory [1,2], the anisotropy introduced at large scales by the forcing (boundary conditions, geometry of an obstacle, etc.) dies out when the energy is transferred down to smaller scales owing to the cascade mechanism (isotropization of the developed turbu-

lence in the inertial range). The analytical results discussed above confirm this classical concept and give a more quantitative picture of the isotropization. The relevance of these results for more realistic situations (scalar advected by the two-dimensional NS field or the turbulent velocity itself) is briefly discussed below.

Effects of compressibility. The anomalous exponents explicitly depend on the parameter $\alpha \geq 0$ that measures the compressibility of the fluid. For the regimes determined by the fixed points F^+ (magnetic model with finite correlation time) and R^A (zero correlation time, with additional inequalities for the parameter \mathcal{A} satisfied by the magnetic case $\mathcal{A} = 1$ and its vicinity), the hierarchy of anisotropic contributions becomes less pronounced as α grows: the anisotropic corrections in Eq. (6.4) become closer to each other and to the leading term as α grows. Thus the compressibility enhances the penetration of the large-scale anisotropy into the inertial range. The situation is opposite for all the other regimes, which arguably can be attributed to the influence of the pressure term.

Effects of pressure. The dependence on the parameter \mathcal{A} , which controls the pressure effects, is essentially different from the dependence on α : for the regimes with nonzero correlation time, the value of the corresponding invariant variable can take only discrete values $\mathcal{A}_* = -1, 1, \alpha$. The behavior for the fixed point $\mathcal{A}_* = 1$, which corresponds to the magnetic case (turbulence without pressure), shows no serious difference from the regimes with pressure. For the rapid-change limit, the exponents continuously depend on \mathcal{A} , and the value of $\mathcal{A} = 1$, where the pressure effects vanish, is not distinguished either.

Relevance for the NS turbulence. The picture outlined above for passively advected fields (a superposition of power laws with universal exponents and nonuniversal amplitudes) seems rather general, being compatible with that established recently in the field of NS turbulence, on the basis of numerical simulations of channel flows and experiments in the atmospheric surface layer; see Refs. [57–62], and references therein. It was shown that the leading terms of the inertial-range behavior are the same for isotropic and anisotropic forcing [57,58]. In the papers [59–62], the velocity correlation functions were decomposed in the irreducible representations of the rotation group. It was argued that in each sector of the decomposition, scaling behavior can be found with apparently universal exponents. The amplitudes of the various contributions are nonuniversal, through the dependence on the position in the flow, the local degree of anisotropy and inhomogeneity, and so on.

This is rather surprising because the equations for the correlation functions in such cases are neither closed nor isotropic and homogeneous. Although the hierarchy similar to Eq. (5.14) is demonstrated by the critical dimensions of certain tensor operators in the stirred NS turbulence, see Sec. 2.3 of Ref. [25], the relationship between them and the anomalous exponents is not obvious there. It is worth recalling here that the so-called “additive fusion rules,” hypothesized for the NS turbulence in a number of papers, Refs. [16,17,26], and characteristic of the models with multifractal behavior (see Ref. [27]), arise naturally in the context of the models of

passive advection owing to their *linearity*. The existing results for the Burgers turbulence can also be interpreted naturally as a consequence of similar fusion rules, where only finite number of dangerous operators contributes to each structure function, see Ref. [28].

One can thus speculate that the anomalous scaling for the genuine turbulence can also appear as a linear phenomenon in the following sense. Let us split the total velocity field into the two parts, the background field and the perturbation (e.g., large-scale and small-scale, or soft and hard components), linearize the original stochastic equation with respect to the latter, choose an appropriate statistics for the former (e.g., Gaussian distribution with Kolmogorov exponents, the description suggested for the large-scale field by the experiment). Then the small-scale perturbation field will show anomalous scaling behavior with nontrivial exponents, which can be calculated systematically within a kind of ε expansion. The corrections due to the nonlinearity can be treated perturbatively, and if they appear irrelevant (e.g., in the sense of Wilson), they will not affect the exponents calculated within the linearized model. In such a case the passive vector field can give the anomalous exponents for the NS velocity field exactly. In other words, such linearized model will belong to the same universality class as the real NS equation, such as the simplified Ising or Heisenberg models are believed to belong to the same universality class as real ferromagnets or binary alloys. It thus might happen that the anomalous behavior of the real inertial-range turbulence is exactly described by one of the nontrivial fixed points for the passive vector model.

Of course, one should not insist too much on such a simple scenario for the anomalous scaling, but it is worthy of attention. In this connection, we could also recall that the passive vector field can indeed reveal the anomalous exponents of the stochastic NS velocity field if the random forcing of the former is chosen to be statistically correlated with that of the latter; see [54].

Validity of the ε expansion and the applicability of the model. A serious question is that of the validity of the ε expansion and the possibility of the extrapolation of the results, obtained within the ε expansions, to the finite values $\varepsilon=O(1)$. For the rapid-change model, the ε expansion works surprisingly well. It was shown [29] that the knowledge of three terms allows one to obtain reasonable predictions for finite $\varepsilon\sim 1$; even the plain ε expansion captures some subtle qualitative features of the anomalous exponents established in analytical and numerical solutions of the exact zero-mode equations and numerical experiments. The agreement can be further improved by using special tricks (such as the “inverse” ε expansion) or interpolation formulas [29].

In the case of the Gaussian model with a finite correlation time, however, there is a natural upper bound for the range of validity of the results, obtained within the ε expansion: for $\varepsilon>1$ the velocity field (and hence all its powers) become dangerous (its critical dimension $\Delta_p=1-\varepsilon$, known exactly due to the Gaussianity, becomes negative). The spectrum of their dimensions is unbounded from below, and in order to find the small- mr behavior one has to sum up all their con-

tributions in the representations such as in Eq. (6.4). This problem is discussed in detail in Ref. [41] for the passive scalar field; the infrared perturbation theory was employed there to perform the required summation for the pair correlation function, in the frozen regime, and within the one-loop approximation for the Wilson coefficients. It was argued that, in that special case, anomalous behavior is described by the same exponent below and above the boundary $\varepsilon=1$, but in general the problem remains open.

Physically, this is a manifestation of the fact that for $\varepsilon>1$, the so-called sweeping effects (kinematic transfer of the small-scale turbulent eddies by the large-scale ones) become important. In a Galilean-covariant problem such composite operators would not give any contribution into the Galilean-invariant quantities (structure functions), as it happens in the RG approach to the stochastic NS equation; see the discussion in Refs. [23,45]. As was pointed out in Ref. [46], the Gaussian model with finite correlation time suffers from the lack of Galilean invariance and therefore misrepresents the sweeping effects: they penetrate into the correlation functions of the scalar and can lead to their strong unphysical dependence on L . Therefore the value $\varepsilon=1$ can also be viewed as the threshold above which the model itself becomes unphysical. (To justify the Gaussian model for $\varepsilon>1$, however, one may recall that the results of Ref. [46] show that it gives a reasonable description of the passive advection in an appropriate frame, where the mean velocity field vanishes.)

We may therefore conclude that the next important step is the analytical derivation of anomalous exponents of a passive scalar and vector quantities advected by the Galilean covariant velocity ensemble, generated by the stochastic NS equation; this work is now in progress.

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